Weakly Exceptional Quotient Singularities

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Doctor of Philosophy
University of Edinburgh
2013

Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text.

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Acknowledgements

I would like to take this opportunity to thank my supervisor Ivan Cheltsov for introducing me to this this topic and for his invaluable explanations and advice. Without his patience and motivation this work would never have been possible.

I wish to thank C. Shramov for useful conversations and helpful comments on the topic. I would also like to thank my thesis committee, Dr. Antony Maciocia and Dr. Vladimir Guletskii, whose comments and suggestions have been of great help in improving my thesis.

Finally, I am forever in debt to my family for all their love and support throughout my life.

Abstract

A singularity is said to be weakly-exceptional if it has a unique purely log terminal blow up. In dimension 2, V. Shokurov proved that weakly exceptional quotient singularities are exactly those of types D_n, E_6, E_7, E_8. This thesis classifies the weakly exceptional quotient singularities in dimensions 3, 4 and 5, and proves that in any prime dimension, all but finitely many irreducible groups give rise to weakly exceptional singularities. It goes on to provide an algorithm that produces such a classification in any given prime dimension.

Lay Summary

In algebraic geometry, we study *varieties*—high-dimensional geometric objects defined via polynomial equations. At most of their points, these objects are *smooth*, i.e. their neighbourhoods look like a lower-dimensional space. However, at other points the variety has a more complicated structure. These points are referred to as the *singularities* of the variety. For example, a singularity can appear when one takes a space, identifies groups of points on it in a regular manner and folds the space, gluing the points in the same group together. Such singularities are called *quotient singularities*.

In order to deal with these singularities, one uses a process called *resolution*, which makes the singularity more simple while exposing some of its structure. However, the resolution of a singularity is not in general unique, and some resolutions turn out to be more useful than others. This means a choice of resolution needs to be made, which can obscure some of the variety's structure. A singularity is called *weakly exceptional* if it has exactly one "good" resolution. This thesis classifies weakly exceptional quotient singularities in low dimensions and proves that in all prime dimensions, "almost all" the quotient singularities are such.

Contents

Chapter 1

Introduction

Let G be a finite subgroup of $\mathrm{GL}_N(\mathbb{C})$. Then G naturally acts on $\mathbb{C}^N$ and on the space of polynomial functions in N variables. Consequently, one can study the quotient variety $V = \mathbb{C}^N/G$ and functions on it. One of the first people who studied this variety was H. Cartan (see [5]), who proved that the singularities of $\mathbb{C}^N/G$ are normal, and so have codimension at least 2. Substantial contributions to the study of these varieties have also been made by C. Chevalley [10], G.C. Shephard and J.A. Todd [35] and D. Prill [27].

The origin of these varieties suggests that they can be looked at from several different angles. On the one hand, the variety is defined by the choice of action of the group G on $\mathbb{C}^N$ (in other words, of the embedding of G into $\mathrm{GL}_N(\mathbb{C})$). Therefore, it is logical to study these varieties by looking at the finite group actions on $\mathbb{C}^N$. On the other hand, for all but the most trivial isomorphism classes of G, these varieties are singular, with the singularity appearing at the image of the origin $0 \ni \mathbb{C}^N$ under the projection $\mathbb{C}^N \to \mathbb{C}^N/G$. The neighbourhood of this quotient singularity (see Definition 2.34) completely defines the entire variety V, so it makes sense to study the variety by studying the singularity on it. The purpose of this thesis is to discuss the correspondence between these two approaches to studying the varieties.

First consider the group-theoretic side of the problem. The first thing to note is that the group G in question is finite, and that any finite group has at most finitely many non-conjugate faithful actions on $\mathbb{C}^N$. Therefore, it makes sense to start by looking at the isomorphism classes of G, rather than at a particular embedding of G into $\mathrm{GL}_N(\mathbb{C})$. Furthermore, it is evident that the properties of V depend on not the group $G \subset \mathrm{GL}_N(\mathbb{C})$ itself, but rather on its image $\bar{G}$ under the natural projection $\mathrm{GL}_N(\mathbb{C}) \to \mathrm{PGL}_N(\mathbb{C})$. This leads to a significant simplification, since it means that given a finite group $\bar{G} \subset \mathrm{PGL}_N(\mathbb{C})$, any lift $G \subset \mathrm{GL}_N(\mathbb{C})$ will lead to the same conclusions. In particular, one can significantly simplify the computations by choosing one of the lifts that are contained in the subgroup $\mathrm{SL}_N(\mathbb{C}) \subset \mathrm{GL}_N(\mathbb{C})$ (see Proposition 2.14). Therefore, one can start with the assumption that G is a finite subgroup of $\mathrm{SL}_N(\mathbb{C})$.

It is also worth asking whether or not the group G is "really" in $\mathrm{SL}_N(\mathbb{C})$, or if the embedding comes from the embedding of G into $\mathrm{SL}_{N'}(\mathbb{C}) \times \mathrm{SL}_{N''}(\mathbb{C})$ (with $N = N' + N''$). Such group actions are called reducible (see Definition 2.4), and can generally be considered as pairs of actions of G on $\mathbb{C}^{N'}$ and $\mathbb{C}^{N''}$. It will be shown (see Theorem 2.45) that for the purposes of this thesis such actions can be disregarded. On the other extreme, one can consider the primitive actions, which can be thought of as "maximally irreducible" (see Definition 2.3). The internal structure of these groups is quite complicated, and there does not seem to be a pattern to the lists of such groups in different dimensions. However, the restriction to the Special Linear groups implies that the number of such groups for any given N is finite (see Lemma 2.16), and there are classifications of all such groups available for small values of N (see Theorem 2.17, Proposition 2.18 and Theorem 2.21). This suggests that in any given dimension, the most interesting case is the one of the irreducible imprimitive group actions. Their structure is (at first glance) relatively easy to understand, but in reality it is very sensitive to the prime factorisation of the number N. It is very easy to construct infinite families of such group actions in any given dimension N, but there is (to my knowledge) no single family that would both have a useful description and encompass all the irreducible imprimitive groups actions. Throughout this thesis, the irreducible imprimitive groups (and the corresponding varieties) will be the ones given most attention.

Now consider the geometric side of the problem. One of the natural ways to study a singularity is to look at its complete and partial resolutions or blowups (see Definition 2.24), or, more precisely, at the exceptional divisors that arise in these resolutions. By doing this, one can deduce certain information about the canonical and anticanonical systems of a neighbourhood of the singularity, as well as about the kinds of metrics the variety admits.

One class of singularities that can be distinguished in this way is the class of exceptional singularities. One says that a singularity $(P \in V)$ is exceptional if for any log canonical boundary (see Definition 2.25), there exists at most one exceptional divisor with discrepancy -1 over P (see Definition 2.26). One distinguishes such singularities because varieties with them have more complicated linear systems $|-n\mathrm{K}_V|$ (for $n \in \mathbb{Z}_{>0}$) than those with non-exceptional singularities. This plays a role in the search for n-complements (certain "good" divisors in these linear systems), which is a part of V. Shokurov's project of studying log flips and the classification of log canonical singularities (see Definition 2.27). However, it turned out that the class of exceptional singularities is fairly small, especially in the quotient singularity case. In particular, for any dimension N, there are only finitely many isomorphism classes of exceptional quotient singularities (see Corollary 2.44).

This prompts one to try to extend this class to include singularities with properties "close enough" to those of exceptional singularities. To do this, one looks at a special class of (partial) resolutions of singularities, called plt blowups. Informally, a plt blowup is a resolution $W \to V$ of $P \in V$ with an irreducible exceptional divisor E and different $\mathrm{Diff}_E(P)$ (see Definition 2.28), such that E is normal and the pair $(E, \mathrm{Diff}_E(P))$ is log terminal (see Definition 2.27). These resolutions do not lead directly to n-complements on V, but they are still very useful in the study of extremal contractions (see [30]) of the singularities. Any

Kawamata log terminal singularity (see Definition 2.27) has such a (partial) resolution, but if it happens to be unique, then this singularity is called weakly exceptional (see Definition 2.32, originally defined in [23]). To justify the choice of the name, it is worth noting that any exceptional singularity is also weakly exceptional (see Lemma 2.33). The concept of weakly exceptional singularities has many applications, including the existence of Kähler-Einstein metrics on compact Kähler manifolds (sometimes referred to as the Calabi problem). This can be seen via the criterion for weak exceptionality stated in the language of Tian's α-invariant (see Theorem 2.40)

Since the same object can be looked at from two different points of view, it is possible to place it into the two relevant class structures. Therefore, it is natural to ask how these two class structures match up. For example, is it possible to say anything about the reducibility of a group action that "induces" a weakly exceptional singularity? What about an exceptional one? Going in the other direction, what (if anything) can be said about the exceptionality of a quotient singularity "induced" by an irreducible group action? Or a primitive one?

The first hints to the answers to these questions can be seen in the two-dimensional example, that was first discussed by V. Shokurov in the paper where the concept of exceptionality was originally proposed (see [36]). From the group-theoretic point of view, the classification of finite subgroups of $\mathrm{SL}_2(\mathbb{C})$ is a classical result, having been known for a very long time. They can be divided into abelian groups $\mathbb{Z}_n$ and $\overline{\mathbb{Z}}_n$, binary dihedral (also known as dicyclic) groups $\overline{\mathbb{D}}_{2n}$ and the three polyhedral groups: the binary tetrahedral ($\overline{\mathbb{A}}_4$), dodecahedral ($\overline{\mathbb{S}}_4$) and icosahedral ($\overline{\mathbb{A}}_5$) groups (for all the notation, see Section 2.1). It is easy to see that the abelian groups act reducibly, the polyhedral groups act primitively, and the binary dihedral groups act irreducibly, but imprimitively (for details, see Theorem 2.17). On the other side, it can be shown (see Theorem 2.22) that the singularities E_6, E_7 and E_8 induced by the polyhedral groups are exceptional, the singularities D_n induced by the binary dihedral groups are weakly exceptional but not exceptional, and the singularities A_n induced by the abelian groups are not weakly exceptional. In fact, since the notions of exceptionality and weak exceptionality were originally defined based on these examples, it makes sense to think of exceptional singularities as higher-dimensional generalisations of 2-dimensional singularities of types E_6, E_7 and E_8. Similarly, the weakly exceptional singularities can be thought of as generalisations of 2-dimensional singularities of types D_n, E_6, E_7 and E_8.

The first guess this example suggests is that a quotient singularity is weakly exceptional if and only if the corresponding group action is irreducible, and that a quotient singularity is exceptional if and only if it is induced by a primitive group action. This is, however, not the case in higher dimensions. Although it is true that every exceptional quotient singularity is induced by a primitive group action (see Proposition 2.43), it is easy to find examples of primitive group actions inducing singularities that are not exceptional or even weakly exceptional. Similarly, a group action inducing a weakly exceptional singularity has to be irreducible, but already in dimension $N = 3$ there are examples of irreducible group actions inducing singularities that are not weakly exceptional. The aim of this thesis is to explore this latter gap.

There has been quite a lot of research done into the topic of exceptional singularities. As mentioned above, a group action that induces an exceptional singularity has to be primitive. Therefore, due to the irregular nature of primitive group actions, it is not realistic to expect a classification of group actions that induce exceptional singularities in arbitrarily high dimensions. On the other hand, these group actions in low dimensions have been completely classified by D. Markushevich, Y. Prokhorov, I. Cheltsov and C. Shramov (in dimensions 3, 4, 5, 6 and 7 — see [25], [7] and [8]). One of the more surprising results is that there are no such quotient singularities in dimension 7 — the conditions for exceptionality turn out to be too strong. It is worth noting that all these classifications have been done in the dimensions where the possible primitive group actions have already been classified from the representation-theoretic point of view. Such classifications of primitive group actions exist up to dimension 11 (as far as I am aware of), and work on processing these with respect to the exceptionality of the induced singularities is currently in progress. However, any developments into higher dimensions will need either new classifications of the group actions or completely new insights into the question of exceptionality.

The class of weakly exceptional singularities has not enjoyed as much popularity. The research into this class of singularities has been concentrated on finding better general criteria for weak exceptionality, rather than on obtaining classification results in any particular dimension. The original idea of the uniqueness of a plt blowup can be observed from looking at the blowup graphs (or their dual graphs) of the two-dimensional weakly exceptional singularities (those of types D and E). Looking at these graphs (see Figures 2.1 and 2.2), one immediately notices that the graphs for the exceptional singularities have a "fork" — i.e. the dual groups have a vertex of degree greater than 2. This property of having a "special" exceptional divisor generalises to the notion of having a unique plt blowup the weak exceptionality was defined by. This idea turned out to be linked to a bound on the α-invariant of Tian (see [40] and [41]) near the singularity, which in turn allowed the discovery of dimension-specific criteria for the singularity being weakly exceptional (see [9] and [7]). However, no classifications of such singularities (outside dimension 2) existed before the work presented in this thesis.

Chapter 2 of this thesis will provide a more precise statement of the results mentioned above. After that, the thesis will set out to construct a classification of low-dimensional weakly exceptional quotient singularities, that relates the group actions in the corresponding dimensions with the exceptionality of the singularities these group actions induce. Since it is known that any group action inducing a weakly exceptional singularity cannot be reducible, only the irreducible group actions need to be considered. With this in mind, it is possible to produce a full classification of group actions inducing weakly exceptional singularities in any given dimension.

In the 3-dimensional case, the list of such group actions (or subgroups of $\mathrm{SL}_3(\mathbb{C})$) is:

Theorem 1.1 (see Theorem 3.3)**.** *Let $G \subset SL_3(\mathbb{C})$ be a finite subgroup. Then G induces a weakly-exceptional but not an exceptional singularity if and only if one of the following holds:*

- *G is an irreducible monomial group, and $\bar{G}$ is not isomorphic to $(\mathbb{Z}_2)^2 \rtimes \mathbb{Z}_3$ or $(\mathbb{Z}_2)^2 \rtimes \mathbb{S}_3$.*
- *G is isomorphic to the group E_{108} of size 108 (see Proposition 2.18).*

Proof: see Section 3.1 □

Since the list of group actions inducing exceptional singularities in this dimension was already known from [25] (see Theorem 3.1), this concludes the exceptionality classification in dimension 3. One thing to note about this classification is that there are only finitely many (in fact, only 3) irreducible group actions that induce singularities that are *not* weakly exceptional.

Moving into higher-dimensional spaces (i.e. higher values of N), the full set of subgroups of $\mathrm{SL}_N(\mathbb{C})$ becomes much too big to be a useful basis for a classification. However, it is just as informative to look at the set of "counterexamples" — the group actions inducing singularities that fail to be weakly exceptional. It is known that a group action has to be irreducible to give rise to a weakly exceptional singularity. Therefore, it is enough to compute the set of irreducible group actions that induces a singularity that still fails to be weakly exceptional. In the 4-dimensional case, the set is as follows:

Theorem 1.2 (see Theorem 3.13)**.** *Let $G \subset SL_4(\mathbb{C})$ be a finite subgroup acting irreducibly. Then the singularity of $\mathbb{C}^4/G$ is not weakly-exceptional exactly when G is conjugate to one of 38 explicitly defined countable (but not all finite) families of groups.*

Proof: see Section 3.2 □

As before, the classification of all 4-dimensional exceptional quotient singularities is known (and can be seen in [7]), so the 4-dimensional classification can be considered complete. It is easy to see that, unlike in the 2- and 3-dimensional cases, this list contains several infinite families of groups. It is also worth noting that the group actions in all these infinite families come from the automorphisms of the variety $\mathbb{P}^1 \times \mathbb{P}^1$.

Continuing with the classification, one can also produce a similar list for the 5-dimensional weakly exceptional quotient singularities:

Theorem 1.3 (see Theorem 3.24)**.** *Let $G \subset SL_5(\mathbb{C})$ be a finite subgroup acting irreducibly. Then the singularity of $\mathbb{C}^5/G$ is weakly-exceptional exactly when:*

> *The action of G is primitive and G contains a subgroup isomorphic to the Heisenberg group of all unipotent 3×3 matrices over $\mathbb{F}_5$ (for a better classification of all such groups, see [26]).*

OR The action of G is monomial (making $G \cong D \rtimes T$, with D an abelian group as above and T a transitive subgroup of $\mathbb{S}_5$), and G does not belong to one of 4 explicitly defined isomorphism classes.

OR The action of G is monomial, and $G \subseteq (\mathbb{Z}_5 \otimes \mathbb{Z}_d^4) \rtimes \mathbb{S}_5 \subset SL_5(\mathbb{C})$ for $d \leq 4$ (note that, up to isomorphism, there are only finitely many such groups G).

Proof: see Section 3.3 □

There are two surprising facts about this list. First of all, it was mildly unexpected to find the cycles of size 11 and 61 as the diagonal parts of the possible monomial groups. These turn out to come from the automorphism groups of Klein cubic and quartic threefolds, which are relatively unknown. I was not able to find much research done into these surfaces, but some additional information can be found in [18].

The other notable fact about this classification is that, unlike in the case of dimension 4, the list is again finite. This means that although not all the irreducible group actions induce weakly exceptional singularities, in dimensions 2, 3 and 5 this gap is as small as can be expected: only finitely many counterexamples appear. Looking at the structure of irreducible imprimitive group actions in various dimensions, one can see that this finiteness is likely to hold in other prime dimensions too. The proof of this conjecture forms the final result of this thesis:

Theorem 1.4 (see Theorem 4.10)**.** *Let q be a positive prime integer. Then there are at most finitely many finite irreducible subgroups $\Gamma \subset SL_q(\mathbb{C})$, such that the singularity induced by Γ is not weakly exceptional.*

Proof: see Section 4.1 □

There are several things that need to be mentioned about the proof of this result. First of all, one can see that the proof works by bounding the maximal possible size of the group, but the bound obtained grows far too fast to be of any use. However, experimental data suggests that this limit is a huge overestimate. This occurs because the bound for the groups size is obtained as a function of the determinant of a certain integer matrix with bounded entries (see Lemma 4.5). The current bound can be improved via a case-by-case analysis of the possible matrices that may appear, but this would not have produced a more enlightening result while making the proof significantly more confusing. Therefore, this analysis was omitted.

The second note is that the proof of this theorem immediately implies an algorithm for computing, for any given prime dimension N, the list of irreducible imprimitive group actions that produce N-dimensional singularities that fail to be weakly exceptional. This algorithm has been described in Section 4.2, and an example of the initial part of it has been implemented as an illustration. Incidentally, this example produces a superset of the set of all 7-dimensional irreducible group actions inducing singularities that fail to be weakly exceptional.

In light of this result, it is natural to ask whether the same (or similar) result can be shown to be true in any other (non-prime) dimensions. Obviously, one can't expect exactly the same result to hold, since it is known that the result is false in dimension 4. Furthermore, there is no hope for dimension 4 to be one of only a few exceptions, since:

Proposition 1.5. *There exist arbitrarily high dimensions N, in which there are infinitely many irreducible subgroups of $SL_N(\mathbb{C})$ inducing weakly exceptional singularities that are not weakly exceptional.*

Proof: Take $N = k^2$, for k as large as desired (and $k > 1$). Write the coordinates of $\mathbb{C}^N$ in a $k \times k$ grid, and consider the result as a $k \times k$ matrix. Consider two finite subgroups $A, B \subset \mathrm{SL}_k(\mathbb{C})$ acting on this matrix as $(a \in A, b \in B)\left(m \in \mathbb{C}^N\right) = amb^T$. Assuming A and B are large enough, this action of $A \times B$ is irreducible. But this action leaves the determinant of the matrix semi-invariant, producing a semi-invariant of degree $k < N$. Therefore, by Theorem 2.47, the singularity induced by this group is not weakly exceptional. Choosing different values for the groups A and B, one can get infinitely many such actions. $\square$

Remark 1.6. *An example of this can be seen in dimension 4 (i.e. $k = 2$) in Lemma 3.12. Similar examples with non-square ($k_1 \times k_2$) grids (if $N = k_1 k_2$, $1 < k_1 < k_2 < N$) can be constructed. It is not known if there are any infinite families of groups that do not appear as a result of such constructions.*

However, there is hope to find some structure to the infinite families that appear on the list. Unfortunately, to do this one would have to look not only at the semi-invariants, but also at the higher-dimensional invariant spaces for the symmetric powers of the group actions. And although it turned out to be possible to bound the size of any cyclic subgroup according to the maximal degree of a semi-invariant (see Corollary 4.6), it is currently not clear how to do something similar with higher-dimensional invariant subspaces.

Another possible extension would be to look for similar classification results over fields $\mathbb{F}_p$ of finite characteristic p (rather than $\mathbb{C}$). Clearly, the finiteness result would be trivial (as $\mathrm{SL}_N(\mathbb{F}_p)$ is a finite group), but it may be possible to say more about the classification of these group in general. However, as far as I know, no progress has been made in that direction so far, so it would be too early to conjecture any possible results.

Chapter 2

Preliminaries

2.1 Notation and terminology

This section will present some notes on the notation used throughout this thesis. This is done to avoid repetitions and ambiguity that can sometimes occur due to different readers being accustomed to different notation. Unless stated otherwise, n is taken to be an arbitrarily chosen integer.

- ζ_n — A primitive n-th root of unity. Any primitive root can be chosen, but the choice needs to be consistent (i.e. $\zeta_{ab}^a = \zeta_b$ for any positive integers a, and b). Sometimes, other symbols (usually, ω) will be used to denote roots of unity. In that case, it will not be assumed that the root is primitive.
- $\mathbb{Z}_n$ — The cyclic group of size n.
- $\mathbb{D}_{2n}$ — The dihedral group of all symmetries of a regular n-gon.
- $\mathbb{S}_n$ — The symmetric group of all permutations of a set of n elements.
- $\mathbb{A}_n$ — The alternating group of all even permutations of a set of n elements.
- $\overline{\mathbb{Z}}_n$, $\overline{\mathbb{D}}_{2n}$, $\overline{\mathbb{S}}_n$, $\overline{\mathbb{A}}_n$ — the central extensions by $\mathbb{Z}_2$ of the groups $\mathbb{Z}_n$, $\mathbb{D}_{2n}$, $\mathbb{S}_n$, $\mathbb{A}_n$ respectively (see, for example, [38, Section 4.4]). Since the generators of these groups will be referred to heavily at the end of Section 3.2, fix the presentations of these groups as:
  - $\overline{\mathbb{Z}}_n = <a_n \mid a_n^{2n} = 1>$
  - $\overline{\mathbb{D}}_{2n} = <a_n, b \mid a_n^n = b^2, b^4 = 1, ba_nb^{-1} = a_n^{-1}>$
  - $\overline{\mathbb{A}}_4 = <[12][34], [14][23], [123]>$, where the basis is chosen so that the two lines spanned by the basis vectors are preserved (setwise) by[12][34] and swapped by [14][23].
  - $\overline{\mathbb{S}}_4 = <[12][34], [14][23], [123], [34]>$, with the subpresentation of $\overline{\mathbb{A}}_4$ as described above.
  - $\overline{\mathbb{A}}_5 = <[12345], [12][34]>$, where the basis is chosen so that [12345] is diagonal.

The last 3 groups are central extensions of permutation groups, and their generators are intentionally named to identify the permutations they correspond to. The relations come from the relations between the permutations.

- $\mathbb{V}_4$ — the Klein group of size 4. Note that the groups $\mathbb{V}_4$, $\mathbb{D}_4$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$ are isomorphic. The notation used for this group will denote the context the group is considered in: $\mathbb{V}_4$ will be used whenever it is considered as a subgroup of $\mathbb{A}_4$ or $\mathbb{S}_4$, and one of the other two will be used whenever it is considered on its own or as a member of a family of cyclic or dihedral groups.

- $A \rtimes B$ is the *semidirect product* of two groups A and B: let β be an action of B on A. Then $A \rtimes B$ is a set of pairs $(a \in A, b \in B)$ with multiplication defined by $(a_1, b_1) \cdot (a_2, b_2) = (a_1\beta(b_1)(a_2), b_1b_2)$. Clearly $A \lhd A \rtimes B$ is a normal subgroup.

- Write "extension of G by scalar elements" (for a finite group G) to mean $H \rtimes G \subset \mathrm{SL}_n(\mathbb{F})$, where H is a (possibly, trivial) subgroup of the centre of $\mathrm{SL}_n(\mathbb{F})$ (i.e. consists of scalar matrices). Note that unless the integer n is prime, there are several non-trivial possibilities for H that give $H \rtimes G$ the same image under the natural projection to $\mathrm{PGL}_n(\mathbb{F})$. Note that the scalar elements are central in $\mathrm{GL}_n(\mathbb{F})$, so this is in fact a central extension.

- $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$— The sets of Natural, Integer, Rational, Real and Complex numbers respectively.

- $\mathrm{GL}_n(\mathbb{C})$, $\mathrm{SL}_n(\mathbb{C})$, $\mathrm{PGL}_n(\mathbb{C})$ — the General Linear, Special Linear and Projective General Linear groups of $n \times n$ matrices over the complex numbers.

- $\mathrm{GL}_n(\mathbb{C})$ is the group of $n \times n$ invertible matrices over the complex numbers. Given a subgroup $G \subset \mathrm{GL}_n(\mathbb{C})$, choose its representation given by the inclusion of G into $\mathrm{GL}_n(\mathbb{C})$. This in turn defines an action (representation) of G on $\mathbb{C}^n$. Unless stated otherwise, this will be the action associated with this group throughout. Furthermore, the name of the group will be used to denote this action when referring to the action's properties (e.g. "group is primitive" will be used to mean that this action of the group is primitive).

- Define the action of a subgroup $G \subset \mathrm{GL}_n(\mathbb{C})$ on polynomials in n variables by, for any polynomial $f(x_1, \ldots, x_n)$ and any matrix (i.e. an element of a group $G \subset \mathrm{GL}_n(\mathbb{C})$) M, setting
$$M(f)(x_1, \ldots, x_n) = f\left(M(x_1, \ldots, x_n)^T\right)^T.$$

- When working with diagonal $n \times n$ matrices, write $[k, a_1, \ldots, a_n]$ for the diagonal $n \times n$ matrix
$$\begin{pmatrix} \zeta_k^{a_1} & & \\ & \ddots & \\ & & \zeta_k^{a_n} \end{pmatrix}.$$

- $\langle A \mid B \rangle$ — The action of a pair of 2×2 matrices on $\mathbb{C}^4$, as described in Section 2.4.

- Say a singularity is *induced by* a group $G \subset \mathrm{GL}_n(\mathbb{C})$ if it is the singularity of the quotient space $\mathbb{C}^n / G$.

2.2 Representations of finite groups

This thesis talks about quotient singularities (see Definition 2.34) — objects obtained as quotients of an affine space by a linear group action. Therefore, it makes sense to first discuss the properties of these actions, namely, the properties of finite dimensional representations of finite groups. This section is not meant to be a comprehensive treatment of the subject, but it will define most of the terms that will be used later on.

Let $N > 1$ be any integer, and consider a finite subgroup $G \subseteq \mathrm{GL}_N(\mathbb{C})$. As stated above, G comes with a choice of action on $\mathbb{C}^N$, which will be the action considered throughout.

Definition 2.1. *Given a group $G \subset GL_N(\mathbb{C})$, a* system of imprimitivity *for G is a set $\{V_1, \dots, V_k\}$ of subspaces of $\mathbb{C}^N$, such that* $\dim V_i > 0$ $\forall i$, *$V_i \cap V_j = \{0\}$ whenever $i \neq j$, $V_1 \otimes \ldots \otimes V_k = \mathbb{C}^N$, and for any $g \in G$ and $i \in \{1, \dots, k\}$, there exists $j(g,i) \in \{1, \dots, k\}$, such that $g(V_i) = V_{j(g,i)}$.*

Remark 2.2. *Clearly, any group $G \subset GL_N(\mathbb{C})$ has at least one system of imprimitivity, namely, $\left\{\mathbb{C}^N\right\}$.*

Definition 2.3. *A group $G \subset GL_N(\mathbb{C})$ is* primitive *if it has exactly one system of imprimitivity.*

Definition 2.4. *A group $G \subset GL_N(\mathbb{C})$ is* irreducible *if for any system of imprimitivity $\{V_1, \dots, V_k\}$ for G, the action of G permutes the subspaces $V_1, \dots, V_k$ transitively.*

Proposition 2.5. *If a group $G \subset GL_N(\mathbb{C})$ with a system of imprimitivity $\{V_1, \dots, V_k\}$ is irreducible, then k divides N, and*

$$\dim V_1 = \dim V_2 = \cdots = \dim V_k = N/k.$$

Proof: Since G is irreducible, given $i, j \in \{1, \dots, k\}$, there exists $g_{i,j} \in G$ such that $g_{i,j}(V_i) = V_j$. Therefore, $\dim V_i = \dim V_j$. Applying this for different pairs (i,j), get $\dim V_1 = \dim V_2 = \cdots = \dim V_k = d$, some $d \in \mathbb{Z}$. Since the pairwise intersections between the V_i-s are trivial, and they span all of $\mathbb{C}^N$, $kd = N$. □

Definition 2.6. *A group $G \subset GL_N(\mathbb{C})$ is* monomial *if there exists a system of imprimitivity $\{V_1, \dots, V_k\}$ for G, such that for all $i \in \{1, \dots, k\}$,* $\dim V_i = 1$.

Proposition 2.7. *Let $G \subset GL_N(\mathbb{C})$ be a finite monomial subgroup. Then have $G \cong D \rtimes T$, where D is abelian, $T \subseteq \mathbb{S}_N$. Given a system of imprimitivity $\{V_1, \dots, V_N\}$ for this group and choosing $0 \neq x_i \in V_i$ for every i, the set*

$\{x_1, \ldots, x_N\}$ *forms a basis for* $\mathbb{C}^N$. *In this basis, every element of* D *is a diagonal matrix, and for every element* $g \in G \setminus D$, *there exists some* $i, j \in \{1, \ldots, N\}$ *with* $i \neq j$ *and* $g(x_i) \in V_j$.

Proof: Since G is monomial, it has at least one system of imprimitivity $\{V_1, \ldots, V_k\}$, such that all the V_i-s have dimension 1. Fix it. Since $V_1, \ldots, V_k$ span $\mathbb{C}^N$, must have $k = N$. The action of G permutes $V_1, \ldots, V_N$, so have a homomorphism $\pi : G \to \mathbb{S}_N$ defined by these permutations. Let $D = \ker(\pi) \trianglelefteq G$ and $T = \mathrm{Im}(\pi) \subseteq \mathbb{S}_N$. Clearly, $G = D \rtimes T$.

For every i, pick a non-zero element $x_i \in V_i$. Since V_i is one-dimensional, x_i spans V_i, and so $\{x_1, \ldots, x_N\}$ is a basis for $\mathbb{C}^N$. Given any $d \in D$, $d(V_i) = V_i$ for every i, and so d must be a diagonal matrix. Therefore, D is abelian.

Let $g \in G$, such that $g(x_i) \in V_i$ for all i. Then $\pi(g)$ is the trivial permutation in $\mathbb{S}_N$, and so $g \in \ker(\pi) = D$. So for any $g \in G \setminus D$, there exist $i \neq j$ with $g(x_i) \in V_j$. $\square$

Proposition 2.8. *Let* $G \subset GL_N(\mathbb{C})$ *be a finite monomial subgroup, and let* $G \cong D \rtimes T$ *be the decomposition from Proposition 2.7. If* G *is irreducible, then* T *is transitive.*

Proof: Assume $T \subseteq \mathbb{S}_N$ is not transitive. Let x_1 be a basis vector from Proposition 2.7. Consider the subspace V of $\mathbb{C}^N$ spanned by $\mathrm{Orb}_G(x_1)$. Since T is not transitive, there exists $j \in \{1, \ldots, N\}$ such that $j \notin \mathrm{Orb}_T(1)$. Therefore, $V_j \cap V = \{0\}$, and so $V \neq \mathbb{C}^N$. However, by construction V must be G-invariant, and so G is not irreducible. $\square$

Remark 2.9. *Note that the reverse implication does not hold: let* $N > 1$ *and take* $D = \{I_N\}$, $T = \mathbb{Z}_N \subseteq \mathbb{S}_N$. *Then* $G \cong \mathbb{Z}_N$ *is abelian, and therefore not irreducible.*

With this decomposition in mind, one can also obtain a much simplified picture of the set of representations of monomial groups:

Lemma 2.10 ([34, §8.1])**.** *If* A *is an abelian normal subgroup of a group* G, *then the degree of each irreducible representation of* G *divides the index* $(G : A)$ *of* A *in* G.

As it will soon become apparent, this thesis is mainly interested in the decompositions of the symmetric powers of the group's representations, in particular, in the G-invariant rings of polynomial functions on $\mathbb{C}^N$:

Definition 2.11. *Given a group* $G \subset GL_N(\mathbb{C})$, *and consider its action on the space* V_d *of degree* d *polynomials of* N *variables. If* $U \subseteq V_d$ *is a* G*-invariant* 1*-dimensional subspace generated by* f, *then* f *is called a* semi-invariant *of* G. *In particular, there exists a homomorphism* $\lambda : G \to \mathbb{C}$, *such that for all* $g \in G$, $g(f) = \lambda(g) f$. *If* $\lambda \equiv 1$ *(and hence* G *fixes* U *pointwise), then* f *is an* invariant *of* G.

This means that it is possible to assume several useful properties of the groups without losing any generality. For instance, consider:

Definition 2.12. *In $GL_N(\mathbb{C})$, a* pseudoreflection *is a diagonal matrix M, such that $N-1$ of the diagonal entries of M are equal to 1, and the remaining one is equal to λ, where $\lambda \neq 1$, but for some $k \in \mathbb{Z}_{\geq 2}$, $\lambda^k = 1$.*

Theorem 2.13 (Chevalley–Shephard–Todd theorem, see [38, Theorem 4.2.5])**.** *The following properties of a finite group G are equivalent:*

- *G is a finite reflection group.*
- *S is a free graded module over S^G with a finite basis.*
- *S^G is generated by n algebraically independent homogeneous elements.*

This means that as far as the properties of rings of functions under the action of G are concerned, one can assume that the group G contains no pseudoreflections. Furthermore, given any finite group $G \in \mathrm{GL}_N(\mathbb{C})$ not containing any pseudoreflections, it is sufficient to look at its image $\bar{G}$ under the natural projection into $\mathrm{PGL}_N(\mathbb{C})$. This means it is possible to decrease the number of groups to be considered via:

Proposition 2.14. *Let $\pi : GL_N(\mathbb{C}) \to PGL_N(\mathbb{C})$ be the natural projection. Then given a finite group $G \subset GL_N(\mathbb{C})$, there exists a group $\hat{G} \subset SL_N(\mathbb{C})$, such that $\pi(G) = \pi(\hat{G})$.*

Proof: Since G is a finite group, choose a minimal set $\{g_1, \ldots, g_k\}$ of generators for G. Since g_i are $N \times N$ matrices, take $d_i \in \mathbb{C}$, such that $d_i^N = \det g_i$ for every i. $G \subset \mathrm{GL}_N(\mathbb{C})$, so $d_i \neq 0\ \forall i$. Take $\hat{g}_i = \frac{1}{d_i} g_i$.

For every i, $\det \hat{g}_i = 1$, and $\pi(g_i) = \pi(\hat{g}_i)$. So, take $\hat{G}$ to be the group generated by $\{\hat{g}_1, \ldots, \hat{g}_j\}$. Then $\hat{G} \subset \mathrm{SL}_N(\mathbb{C})$, and $\pi(G) = \pi(\hat{G})$. □

It will be shown later in the thesis that, for the purposes of this thesis, one is interested purely in $\pi(G)$, and not G itself. Therefore, one can without loss of generality assume that $G \subset \mathrm{SL}_N(\mathbb{C})$. This assumption will be made throughout the thesis.

Remark 2.15. *Since $G \subset SL_N(\mathbb{C})$, G contains no pseudoreflections.*

Proof: The determinant of a pseudoreflection matrix is not equal to 1. □

This assumption also implies a very useful limitation on the possible groups G, via the following classical theorem:

Lemma 2.16 (Jordan's theorem — see, for example, [17])**.** *For any given N, there are only finitely many finite primitive subgroups of $SL_N(\mathbb{C})$.*

This means that all the primitive groups in any given dimension can be classified (and, in fact, listed). Such classifications are known in low dimensions: I am aware of the classifications up to dimension 9 and in dimension 11. Some authors mention that the classification for dimension 10 also exists, but I have been unable to find any paper or series of papers that would provide (or immediately imply) the list.

The classification in dimension 2 is a well-known classical result:

Theorem 2.17. *Let $G \subset SL_2(\mathbb{C})$ be a finite group, Let $\bar{G}$ be its image under the projection to $PGL_2(\mathbb{C}) = Aut(\mathbb{P}^1)$. Then $\bar{G}$ belongs to one of the following classes:*

- *Cyclic:* $\mathbb{Z}_n$, $n \geq 1$.
- *Dihedral:* $\mathbb{D}_{2n} = <a, b \mid a^n = b^2 = id,\ bab = a^{-1}>$ *($n \geq 2$).*
- *Polyhedral groups* $\mathbb{A}_4$, $\mathbb{S}_4$, $\mathbb{A}_5$.

Lifting the actions of these groups to $SL_2(\mathbb{C})$, one sees that $\bar{G}$ must be conjugate to one of the following:

- *Binary cyclic group*

  $$\overline{\mathbb{Z}}_n = <a \mid a^{2n} = 1>$$

  All its faithful representations are 1–dimensional, and are of the form of $a \rightsquigarrow \zeta_{2n}^l$, some $l \in \mathbb{Z}$. Thus a 2–dimensional representation has to be a direct sum of two such.

- *Cyclic group $\mathbb{Z}_n$, where n is odd. Similar to the binary cyclic group, this group is abelian, and so its two-dimensional representation must be a direct sum of two one-dimensional representations. When one projects these groups into $PGL_2(\mathbb{C})$, the kernel is trivial, and a lift of the projection back to $SL_2(\mathbb{C})$ can be chosen to be either $\mathbb{Z}_n$ or $\overline{\mathbb{Z}}_n$. To simplify notation later on, always choose to lift it as $\overline{\mathbb{Z}}_n$.*

- *Binary dihedral group*

  $$\overline{\mathbb{D}}_{2n} = <a, b \mid a^n = b^2, b^4 = 1, aba^{-1} = a^{-1}>$$

  The suitable 2–dimensional representations of this group are indexed by different choices of ζ_{2n}. They are:

  $$a \rightsquigarrow \begin{pmatrix} \zeta_{2n} & 0 \\ 0 & \zeta_{2n}^{-1} \end{pmatrix},\ b \rightsquigarrow \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

- *Binary tetrahedral group*

  $$\overline{\mathbb{A}}_4 = <\zeta_4(12)(34), \zeta_4(14)(23), \zeta_4(123)>$$

  (using standard notation for elements of the symmetric group). Similarly to above, the suitable 2–dimensional representations of this group are determined by the choice of ζ_8. They are:

  $$\zeta_4(12)(34) \rightsquigarrow \begin{pmatrix} \zeta_8^2 & 0 \\ 0 & -\zeta_8^2 \end{pmatrix},\ \zeta_4(14)(23) \rightsquigarrow \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

  $$\zeta_4(234) \rightsquigarrow \frac{1}{\sqrt{2}} \begin{pmatrix} \zeta_8^7 & \zeta_8^7 \\ \zeta_8^5 & \zeta_8 \end{pmatrix}$$

- *Binary octahedral group*

$$\overline{\mathbb{S}}_4 = <\zeta_4(12)(34), \zeta_4(14)(23), \zeta_4(123), \zeta_4(34)>$$

This group only has 2 suitable representations, each having a subrepresentation isomorphic to the representation of $\overline{\mathbb{A}_4}$ that uses the same value of ζ_8. The extra generator acts as

$$\zeta_4(34) \rightsquigarrow \begin{pmatrix} 0 & \zeta_8 \\ -\zeta_8^7 & 0 \end{pmatrix}$$

- *Binary icosahedral group*

$$\overline{\mathbb{A}}_5 = <\zeta_4(12345), \zeta_4(12)(34)>$$

$$\zeta_4(12345) \rightsquigarrow \begin{pmatrix} \zeta_5^3 & 0 \\ 0 & -\zeta_5^2 \end{pmatrix},$$

$$\zeta_4(12)(34) \rightsquigarrow \frac{1}{\sqrt{5}} \begin{pmatrix} -\zeta_5 + \zeta_5^4 & \zeta_5^2 - \zeta_5^3 \\ \zeta_5^2 - \zeta_5^3 & \zeta_5 - \zeta_5^4 \end{pmatrix}$$

One can see that these group actions are of the following types:

- *The actions of cyclic groups are not irreducible.*
- $\overline{\mathbb{A}}_4$, $\overline{\mathbb{S}}_4$, $\overline{\mathbb{A}}_5$ *have primitive actions*
- *Binary dihedral groups have imprimitive monomial actions.*

Proof: The list of the isomorphism classes is a classical result, that can be attributed to F. Klein (or Plato). A modern treatment can be found in [13]. Explicit matrix representations can easily be computed by hand. □

A similar result in dimension 3 is also well-known and can be found in, for example, in [2]. However, the classical exposition of this result is misleading, often leading to two of the groups being missed out (actually, direct product of two other groups in the list and $\mathbb{Z}_3$, the center of $\mathrm{SL}_3(\mathbb{C})$). Therefore, a modern (and more explicit) statement of the same result will be used here:

Proposition 2.18 (see [42, Theorem A])**.** *Define the following matrices:*

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix} \quad T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad W = \begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega \end{pmatrix}$$

$$U = \begin{pmatrix} \epsilon & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & \epsilon\omega \end{pmatrix} \quad Q = \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & b \\ 0 & c & 0 \end{pmatrix} \quad V = \frac{1}{\sqrt{-3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}$$

where $\omega = e^{2\pi i/3}$, $\epsilon^3 = \omega^2$ and $a, b, c \in \mathbb{C}$ are chosen arbitrarily, as long as $abc = -1$ and Q generates a finite group.

Up to conjugacy, any finite subgroup of $SL_3(\mathbb{C})$ belongs to one of the following types:

1. *Diagonal abelian group.*
2. *Group isomorphic to an irreducible finite subgroup of $GL_2(\mathbb{C})$, preserving a plane $\mathbb{C}^2 \subset \mathbb{C}^3$, and not conjugate to a group of type (1).*
3. *Group generated by the group in (1) and T and not conjugate to a group of type (1) or (2).*
4. *Group generated by the group in (3) and Q and not conjugate to a group of types (1)—(3).*
5. *Group E_{108} of size 108 generated by S, T and V.*
6. *Group F_{216} of size 216 generated by the group E_{108} in (5) and an element $P := UVU^{-1}$.*
7. *Hessian group $\mathbb{H}_{648}$ of size 648 generated by the group E_{108} in (5) and U.*
8. *Simple group of size 60 isomorphic to alternating group $\mathbb{A}_5$.*
9. *Klein's simple group $\mathbb{K}_{168}$ of size 168 isomorphic to permutation group generated by (1234567), $(142)(356)$, $(12)(35)$.*
10. *Group of size 180 generated by the group $\mathbb{A}_5$ in (8) and W.*
11. *Group of size 504 generated by the group $\mathbb{K}_{168}$ in (9) and W.*
12. *Group G of size 1080 with its quotient $G/ <W>$ isomorphic to the alternating group $\mathbb{A}_6$.*

Remark 2.19. *The groups often missed when working from the classical sources are those labelled "10" and "11".*

These groups can be put into the form of the algebraic classification of groups given in the earlier sections as follows:

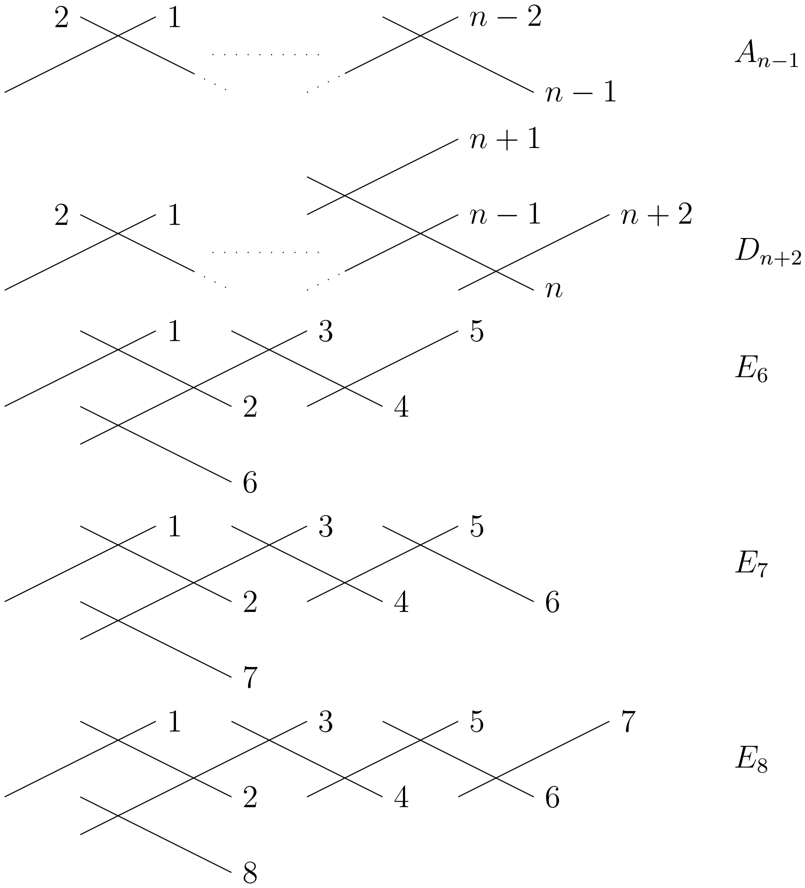
Remark 2.20. *The list of groups in Proposition 2.18 can be subdivided as follows:*

- *Groups of types (1) and (2) are not irreducible.*
- *Groups of types (5)—(12) are primitive.*
- *Groups of types (3) and (4) are irreducible and monomial.*

Proof: Immediate from the lists of generators given above and in [42]. □

Since 4 is not a prime number, the structure of finite imprimitive subgroups of $\mathrm{SL}_4(\mathbb{C})$ is much more complicated than that of analogous subgroups in lower dimensions. For the structure of the imprimitive subgroups in $\mathrm{GL}_4(\mathbb{C})$, one can refer to [16] and [19], and the restriction to the subgroups of $\mathrm{SL}_4(\mathbb{C})$ does not lead to any significant simplifications. Furthermore, even in prime dimensions, the list of all imprimitive groups becomes too large to effectively process. However, the lists of primitive groups still remain fairly short and manageable:

Figure 2.1: Blowup graphs of 2-dimensional singularities



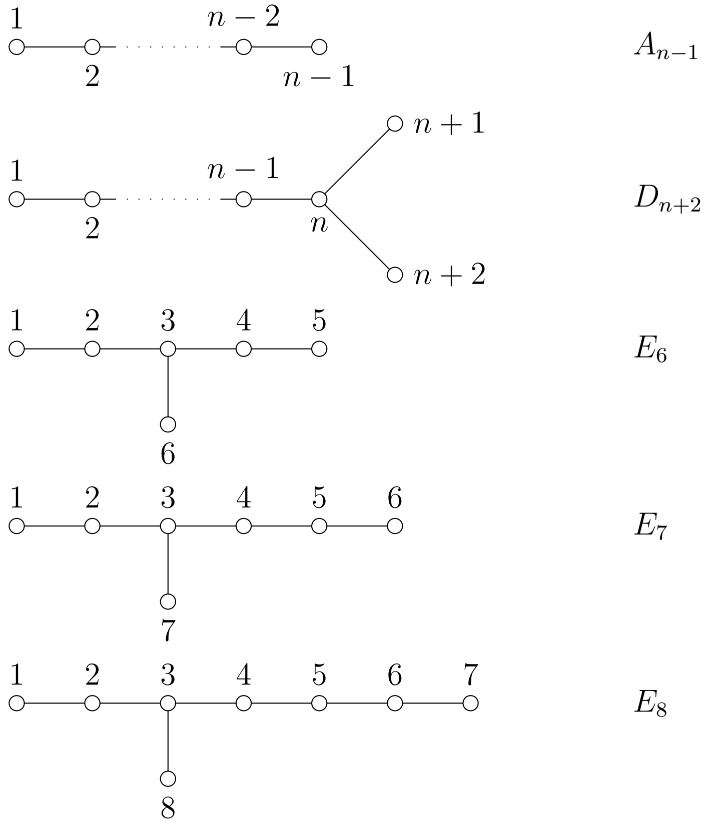
Theorem 2.21 ([15, §8.5]). *If $G \subset SL_5(\mathbb{C})$ is a finite group acting primitively, then either G is one of $\mathbb{A}_5$,$\mathbb{A}_6$, $\mathbb{S}_5$, $\mathbb{S}_6$, $PSL_2(11)$ and $Sp_4(\mathbb{F}_3)$. or G is a subgroup of the normalizer $\mathbb{HM}$ of the Heisenberg group $\mathbb{H}$ of all unipotent 3×3 matrices over $\mathbb{F}_5$, such that $\mathbb{H} \subset G \subseteq \mathbb{HM}$.*

2.3 Exceptional and weakly exceptional singularities

The topic of this thesis was inspired by the following example:

Example 2.22 (see [37, Section 5.2.3]). *Consider the varieties $\mathbb{C}^2/G$, where $G \subset SL_2(\mathbb{C})$ is a finite subgroup. The list of such subgroups is well-known (see Theorem 2.17), and these varieties give the usual examples of the well-known A-D-E singularities: the groups $\overline{\mathbb{Z}}_n$ (or $\mathbb{Z}_n$ if n is odd), $\overline{\mathbb{D}}_{2n}$, $\overline{\mathbb{A}}_4$, $\overline{\mathbb{S}}_4$ and $\overline{\mathbb{A}}_5$ correspond to singularities of types A_{n-1}, D_{n+2}, E_6, E_7 and E_8 respectively. One can blow these singularities up and look at their blowup graphs. These graphs can be seen in Figure 2.1.*

Figure 2.2: Dual blowup graphs of 2-dimensional singularities



However, it is easier to think about the dual blowup graphs of these singularities (where the vertices are the exceptional divisors, and the edges signify their intersection). The dual blowup graphs for the five singularity types can be seen in Figure 2.2.

One immediately observes that unlike the graph for type A singularities, the graphs for the singularities of types D and E all have a "fork" in them (i.e. a vertex of degree 3*). The singularities with such a "fork" have been named* weakly exceptional.

In order to generalise this, one first needs to define some terminology:

Definition 2.23. *Let X and Y be normal varieties, and $f : Y \to X$ a projective morphism, such that $f_*\mathcal{O}_Y = \mathcal{O}_X$. Then f is called either a* contraction *from Y to X or an* extraction *from X to Y.*

Definition 2.24. *A* blowup *is a birational extraction. Two blowups $f : Y \to X$ and $f' : Y' \to X$ are* birationally equivalent *if the rational map $f^{-1} \circ f'$ is an isomorphism in codimension one.*

Definition 2.25. *Let X be a variety, and $D = \sum d_i D_i$ a divisor on X, with D_i distinct prime Cartier divisors and $d_i \in \mathbb{R}$. Then D is a* boundary *if for all i, $0 \leq d_i \leq 1$. D is a* subboundary *if there exists a boundary $D' = \sum d_i' D_i$ with $0 \leq d \leq d' \leq 1$ for every i.*

Definition 2.26. *Let $Z \subseteq X$, assume X only has normal singularities along Z, and $D = \sum d_i D_i$ is a boundary on X ($d_i \in \mathbb{R}$, D_i prime divisors). Let $f : Y \to X$ be a blowup, $E = \cup E_i$ its exceptional divisor, and $\hat{D}$ the proper transform of D on Y. Then f is a* log resolution *of the pair (X, D) along Z if:*

- *Y is nonsingular near $f^{-1}(Z)$*
- *Supp $\left(\hat{D}\right) \cup E$ is a normal crossing divisor on Y near $f^{-1}(Z)$.*

Writing

$$K_Y \equiv f^* (K_X + D) + \sum_{E_i} a (E_i, X, D) E_i$$

with E_i running over the prime divisors on Y, the numbers $a (E_i, X, D) \in \mathbb{R}$ are called discrepancies *of f (or of the pair (X, D)). For each component $\hat{D}_i$ of $\hat{D}$, $a\left(\hat{D}_i, X, D\right) = -d_i$.*

Definition 2.27. *Let $Z \subseteq X$, assume X only has normal singularities along Z, and $D = \sum d_i D_i$ is a boundary on X ($d_i \in \mathbb{R}$, D_i prime divisors). The pair (X, D) (or, abusing notation, $K_X + D$) is* log canonical *along Z if*

- *$K_X + D$ is $\mathbb{R}$-Cartier.*
- *For any blowup $f : Y \to X$, writing*

  $$K_Y \equiv f^* (K_X + D) + \sum_{E} a_f (E, X, D) E$$

  with E running over the prime divisors on Y, produces $a_f (E, X, D) \geq -1$ for every E near $f^{-1}(Z)$.

Furthermore, a log canonical pair is:

- *Purely log terminal (plt) along Z if $a_f (E, X, D) > -1$*
- *Kawamata log terminal (klt) along Z if $a_f (E, X, D) > -1$ and $d_i \neq 1$ $\forall i$*
- *Canonical along Z if $a_f (E, X, D) \geq 0$*
- *Terminal along Z if $a_f (E, X, D) > 0$*

(for all such f and for every prime E near $f^{-1}(Z)$). A singularity $(P \in X)$ is said to be log canonical, plt, klt, canonical or terminal if it has the relevant properties with $Z = P$ and D trivial.

Definition 2.28. *Let X be a normal variety and let $S + B$ be a boundary on X, where $S = \lfloor S + B \rfloor \neq \emptyset$ and $B = \{S + B\}$. Assume that $K_X + S + B$ is log canonical in codimension 2. Then the* different *of B on S is defined by*

$$K_S + \mathit{Diff}_S (B) \equiv (K_X + S + B)|_S .$$

Now the example above can be generalised to higher dimension by the following:

Definition 2.29. *Let $(V \ni O)$ be a germ of a Kawamata log terminal singularity. The singularity is said to be* exceptional *if for every effective $\mathbb{Q}$-divisor D_V on the variety V, such that the log pair (V, D_V) is log canonical, there exists at most one exceptional divisor over the point O with discrepancy -1 with respect to the pair (V, D_V).*

Theorem 2.30 (see [7, Theorem 3.7])**.** *Let $(V \ni O)$ be a germ of a Kawamata log terminal singularity. Then there exists a birational morphism $\pi : W \to V$ such that the following hypotheses are satisfied:*

- *the exceptional locus of π consists of one irreducible divisor E such that $O \in \pi(E)$,*
- *the log pair (W, E) has purely log terminal singularities.*
- *the divisor $-E$ is a π-ample $\mathbb{Q}$-Cartier divisor.*

Definition 2.31 ([23])**.** *Let $(V \ni O)$ be a germ of a Kawamata log terminal singularity, and $\pi : W \to V$ be a birational morphism satisfying the conditions of Theorem 2.30. Then π is a* plt blow-up *of the singularity.*

This naturally leads to the following definition:

Definition 2.32 ([23])**.** *We say that the singularity $(V \ni O)$ is* weakly-exceptional *if it has a unique plt blow-up.*

To justify the latter name, it is useful to note that:

Lemma 2.33 (see [28, Theorem 4.9])**.** *If $(V \ni O)$ is exceptional, then $(V \ni O)$ is weakly exceptional.*

Looking at the famous $A-D-E$ classification of 2-dimensional singularities, it is easy to see that the exceptional singularities are generalisations of singularities of type E, while the weakly exceptional ones are generalisations of singularities of types D and E.

Definition 2.34. *The singularity $O \ni V$ on an N-dimensional complex variety V is a* quotient singularity *if there exists an affine open neighbourhood $U \subset V$ of O and a finite group $G \subseteq GL_N(\mathbb{C})$, such that U is isomorphic to an open neighbourhood of the singularity of $\mathbb{C}^N/G$.*

In dimension 2, all the singularities considered in Example 2.22 are quotient singularities, so for the purposes of this thesis, we focus on the quotient singularity case in higher dimensions too. This assumption immediately implies two things that will greatly simplify the study of such singularities. First of all, it implies that, at least locally, the classification of such singularities is closely tied to the classification of finite subgroups of the special linear groups of the relevant dimension (see section 2.2). Secondly, this gives a predictable action of a group G on an affine neighbourhood of the singularity. This makes it natural to use $\bar{G}$-invariant numeric invariants to study this singularity.

Definition 2.35 (see [40] and [41]). *Let X be a smooth Fano variety (see [21]) of dimension n, and let $g = (g_{ij})$ be a Kähler metric, such that*

$$\omega = \frac{\sqrt{-1}}{2\pi} \sum g_{ij} dz_i \wedge d\bar{z}_j \in c_1(X)$$

Let $\bar{G} \subseteq Aut(X)$ be a compact subgroup, such that g is $\bar{G}$-invariant. Let $P_{\bar{G}}(X, g)$ be the set of C^2 smooth $\bar{G}$-invariant functions $X \to \mathbb{C}$, such that $\forall \phi \in P_{\bar{G}}(X, g)$,

$$\omega + \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\phi > 0$$

and $\sup_X \phi = 0$. Then the $\bar{G}$-invariant α-invariant *of X is*

$$\alpha_{\bar{G}}(X) = \sup \left\{ \lambda \in \mathbb{Q} \;\middle|\; \begin{array}{c} \exists C \in \mathbb{R}, \text{ such that } \forall \phi \in P_{\bar{G}}(X, g), \\ \int_X e^{-\lambda\phi}\omega^n \leq C \end{array} \right\}$$

where n is the dimension of X.

Definition 2.36. *Let X be a variety with at most Kawamata log terminal singularities (see [22, Definition 3.5]) and D an effective $\mathbb{Q}$-divisor on X. Let $Z \subseteq X$ be a closed non-empty subvariety. Then the* log canonical threshold *of D along Z is*

$$c_Z(X, D) = \sup \{ \lambda \in \mathbb{Q} \mid \text{the pair } (X, \lambda D) \text{ is log canonical along } Z \}$$

To simplify notation, write $c_X(X, D) = c(X, D)$.

There exists an equivalent complex analytic definition of the log canonical threshold:

Proposition 2.37 (see [22, Proposition 8.2]). *Let X be a smooth complex variety, Z a closed non-empty subscheme of X, and f a non-zero regular function on X. Then*

$$c_Z(X, \{f = 0\}) = \sup \left\{ \lambda \in \mathbb{Q} \;\middle|\; \|f\|^{-\lambda} \text{ is locally } L^2 \text{ near } Z \right\}$$

(where $\|\cdot\|$ denotes the standard norm on $\mathbb{C}$).

Definition 2.38. *Let G be a finite subgroup of $GL_N(\mathbb{C})$, where $N \geq 2$, and let $\bar{G}$ be its image under the natural projection into $PGL_N(\mathbb{C})$. Then the* global $\bar{G}$-invariant log canonical threshold *of $\mathbb{P}^{N-1}$ is:*

$$\operatorname{lct}\left(\mathbb{P}^{N-1}, \bar{G}\right) = \inf \left\{ c\left(\mathbb{P}^{N-1}, D\right) \;\middle|\; \begin{array}{l} D \text{ is a } \bar{G}\text{-invariant effective} \\ \mathbb{Q}\text{-divisor on } \mathbb{P}^{N-1}, \text{ such that} \\ D \sim_{\mathbb{Q}} -K_{\mathbb{P}^{N-1}} \end{array} \right\}$$

Remark 2.39 (see [12, Theorem A.3]). $\operatorname{lct}\left(\mathbb{P}^{N-1}, \bar{G}\right) = \alpha_{\bar{G}}\left(\mathbb{P}^{N-1}\right)$

Using these invariants produces less abstract criteria for the weak exceptionality of a given singularity:

Theorem 2.40 (see [7, Theorem 3.15])**.** *The singularity $\mathbb{C}^N/G$ is weakly exceptional if and only if* $\mathrm{lct}(\mathbb{P}^{N-1}, \bar{G}) \geq 1$*.*

Remark 2.41. *A similar condition is often necessary in order to compute conjugacy classes in higher-dimensional Cremona groups. For details, see [6].*

Corollary 2.42. *If $H \subseteq G \subset GL_N(\mathbb{C})$ and the singularity of $\mathbb{C}^N/H$ is weakly exceptional, then so is the singularity of $\mathbb{C}^N/G$.*

Since the singularities are induced by actions of finite linear groups, it is natural to try to see how the properties of the actions relate to the exceptionality of the singularities.

Proposition 2.43 (see [29, Proposition 2.1])**.** *Let $G \subset SL_N(\mathbb{C})$ be a finite subgroup that induces an exceptional singularity. Then G is primitive.*

It follows, that

Corollary 2.44. *For any given N, only finitely many finite subgroups of $SL_N(\mathbb{C})$ induce exceptional singularities.*

Proof: Immediate by Proposition 2.43 and Jordan's theorem (Theorem 2.16) □

This suggests that the class of exceptional quotient singularities is rather small. Indeed, it has been proven that in some dimensions, no quotient singularities are exceptional (see [8]). On the other hand, even in dimension 2, there is an infinite family of weakly exceptional singularities. This thesis is devoted to studying this bigger class of singularities.

Theorem 2.45. *Let $G \subset SL_N(\mathbb{C})$ be a finite subgroup that induces a weakly-exceptional singularity. Then G is irreducible.*

Proof: The argument is similar to that in [29, Proposition 2.1]. □

In order to classify WE singularities, it would be very useful to have a group-theoretic criterion for the group action that induces it. Looking at the two-dimensional case, the first guess would be irreducibility. However, this proves to be insufficient, since:

Example 2.46. *Consider the group $G = (\mathbb{Z}_2)^2 \rtimes \mathbb{S}_3$. This group has an irreducible action on $\mathbb{C}^3$, but the singularity it induces is not weakly exceptional (see Theorem 3.3).*

The next guess can be suggested by the following:

Theorem 2.47. *Let $G \subset SL_N(\mathbb{C})$ be a finite subgroup with a semi-invariant of degree $d < N$. Then the singularity G induces is not weakly exceptional.*

Proof: Let f_d be such a semi-invariant. Let $S = \{f_d = 0\} \subset \mathbb{P}^{N-1}$, and consider the divisor $D = \frac{N}{d}S$. Then $D \sim_{\mathbb{Q}} -\mathrm{K}_{\mathbb{P}^{N-1}}$ is a $\bar{G}$-invariant effective $\mathbb{Q}$-divisor, and $c\left(\mathbb{P}^{N-1}, D\right) \leq \frac{d}{N}$. Thus, since $d < N$,

$$\mathrm{lct}\left(\mathbb{P}^{N-1}, \bar{G}\right) \leq c\left(\mathbb{P}^{N-1}, D\right) \leq \frac{d}{N} < 1$$

Therefore, the singularity G induces is not weakly exceptional by Theorem 2.40. $\square$

In dimension 2, this implies the group action has no degree 1 semi-invariants, which is actually equivalent to irreducibility. Furthermore, this proves to be a sufficient condition in dimension 3:

Theorem 2.48 (see [7, Theorem 3.18])**.** *Suppose that $G \subset SL_3(\mathbb{C})$ is a finite group and $\bar{G}$ is its image under the natural projection into $PGL_3(\mathbb{C})$. Then the following are equivalent:*

- *the inequality* $\operatorname{lct}(\mathbb{P}^2, \bar{G}) \geq 1$ *holds,*
- *the group G does not have semi-invariants of degree at most 2.*

This begs the question: Is this a sufficient condition in general? Sadly, the answer is negative:

Theorem 2.49 (see [7, Theorem 4.1])**.** *Suppose that $G \subset SL_4(\mathbb{C})$ is a finite group and $\bar{G}$ is its image under the natural projection into $PGL_4(\mathbb{C})$. Then the inequality* $\operatorname{lct}(\mathbb{P}^3, \bar{G}) \geq 1$ *holds if and only if the following conditions are satisfied:*

- *the group G is irreducible,*
- *the group G does not have semi-invariants of degree at most 3,*
- *there is no $\bar{G}$-invariant smooth rational cubic curve in $\mathbb{P}^3$.*

Remark 2.50. *The last condition is necessary*

Proof: Post-factum, this can be seen from Proposition 3.6 $\square$

On the other hand, the semi-invariants become a sufficient condition again in dimension 5:

Theorem 2.51 ([9])**.** *Suppose that $G \subset SL_5(\mathbb{C})$ is a finite group and $\bar{G}$ is its image under the natural projection into $PGL_5(\mathbb{C})$. Then the following are equivalent:*

- *the inequality* $\operatorname{lct}(\mathbb{P}^4, \bar{G}) \geq 1$ *holds,*
- *the group G does not have semi-invariants of degree at most 4.*

This raises the question of whether or not the lack of low-degree semi-invariants is a sufficient condition in some higher dimension. It will be shown in Chapter 4, Lemma 4.3, that it indeed is for imprimitive groups in all prime dimensions. However, as seen from Remark 2.50, that this is not the case in composite dimensions.

In general dimension, the following holds:

Theorem 2.52 ([9, Theorem 1.12])**.** *Let G be a finite group in $GL_{n+1}(\mathbb{C})$ that does not contain reflections. If $\mathbb{C}^{n+1}/G$ is not weakly exceptional, then there* *is a $\bar{G}$-invariant, irreducible, normal, Fano type projectively normal subvariety $V \subset \mathbb{P}^n$ such that*

$$\deg V \leq \left(\begin{array}{c} n \\ \dim V \end{array} \right)$$

and for every $i \geq 1$ and for every $m \geq 0$ one has

$$h^i \left(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n} (\dim V + 1) \otimes \mathcal{I}_V\right) = h^i \left(V, \mathcal{O}_V (m)\right) = 0,$$

$$h^0 \left(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n} (\dim V + 1) \otimes \mathcal{I}_V\right) \geq \left(\begin{array}{c} n \\ \dim V + 1 \end{array} \right),$$

where $\mathcal{I}_V$ is the ideal sheaf of the subvariety $V \subset \mathbb{P}^n$. Let Π be a general linear subspace of $\mathbb{P}^n$ of codimension $k \leq \dim V$. Put $X = V \cap \Pi$. Then $h^i (\Pi, \mathcal{O}_\Pi (m) \otimes \mathcal{I}_X) = 0$ for every $i \geq 0$ and $m \geq k$, where $\mathcal{I}_X$ is the ideal sheaf of the subvariety $X \subset \Pi$. Moreover, if $k = 1$ and $\dim V \geq 2$, then X is irreducible, projectively normal, and $h^i (X, \mathcal{O}_X (m)) = 0$ for every $i \geq 1$ and $m \geq 1$.

2.4 Miscellaneous material

This section contains assorted results and constructions that I felt belong in this chapter, but are not directly related to the two major topics covered above. Some of them are difficult results from various areas of mathematics, in which case, relevant references will be given. Others may be fairly simple results that I felt needed to be explicitly proven for the sake of completeness. The motivation for including the results will not be given here, but instead will become clear later in the text, when these results are used.

Theorem 2.53 ([24])**.** *Given $m \in \mathbb{Z}$, $m > 1$, let $W(m)$ be the set of integers $n \geq 0$, for which there exist $\omega_1, \ldots, \omega_n \in \mathbb{C}$ with $\omega_i^m = 1$ $\forall i$ and $\omega_1 + \cdots + \omega_n = 0$. Take $m = p_1^{a_1} \cdots p_r^{a_r}$ the prime decomposition of m. Then*

$$W(m) = \mathbb{N}p_1 + \mathbb{N}p_2 + \cdots + \mathbb{N}p_r$$

Definition 2.54 ([11])**.** *An n-by-n matrix M is called* circulant *if it is of the form*

$$M = \begin{pmatrix} a_1 & a_2 & \cdots & a_{n-1} & a_n \\ a_n & a_1 & \cdots & a_{n-2} & a_{n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_3 & a_4 & \cdots & a_1 & a_2 \\ a_2 & a_3 & \cdots & a_n & a_1 \end{pmatrix}$$

for some numbers $a_1, \ldots, a_n \in \mathbb{C}$.

Lemma 2.55 ([11, §3.2])**.** *For any circulant matrix M with $a_1, \ldots, a_n$ as above and any $\omega \in \mathbb{C}$ with $\omega^n = 1$, M has an eigenvector $v = (1, \omega, \omega^2, \ldots, \omega^{n-1})^T$ with eigenvalue $\lambda = \sum_{i=1}^n a_i \omega^{i-1}$. All the eigenvalues of M are of this form.*

Proof: It is easy to check that vectors of this form are indeed eigenvectors of M with relevant eigenvalues. These form a set of n linearly independent eigenvectors (can be seen via Theorem 2.53), so these are all the possible eigenvalues and eigenvectors for M. $\square$

Theorem 2.56. *Let $G \subseteq \mathbb{S}_5$ be a subgroup, such that*

$$\mathbb{Z}_5 = <(1\ 2\ 3\ 4\ 5)> \subseteq G \subseteq \mathbb{S}_5.$$

Then G is isomorphic to one of:

- $\mathbb{Z}_5 = <(1\ 2\ 3\ 4\ 5)>$,
- $\mathbb{D}_{10} = <(1\ 2\ 3\ 4\ 5), (2\ 5)(3\ 4)>$,
- $\mathbb{GA}(1,5) = <(1\ 2\ 3\ 4\ 5), (2\ 3\ 5\ 4)>$,
- $\mathbb{A}_5 = <(1\ 2\ 3\ 4\ 5), (1\ 2\ 3)>$,
- $\mathbb{S}_5 = <(1\ 2\ 3\ 4\ 5), (1\ 2)>$,

where the generators are written using the standard notation for elements of permutation groups. Furthermore,

$$\mathbb{Z}_5 \subset \mathbb{D}_{10} \subset \mathbb{A}_5, \ \mathbb{GA}(1,5) \subset \mathbb{S}_5$$

The groups $\mathbb{A}_5$ and $\mathbb{GA}(1,5)$ are not subgroups of each other.

Proof: By enumerating all possible additional generators. See, for example, appendix of [39]. The inclusions can be seen from the listed generators. $\square$

Theorem 2.57 ([20])**.** *Let S be a smooth cubic surface, and $G \subseteq Aut(S)$ a finite subgroup. Then G is isomorphic to one of the following groups:*

- $\{Id_G\}$, *the trivial group.*
- $\mathbb{Z}_2$.
- $\mathbb{Z}_4$.
- $\mathbb{Z}_8$.
- $(\mathbb{Z}_2)^2$.
- $\mathbb{S}_3$.
- $\mathbb{S}_3 \times \mathbb{Z}_2$.
- $\mathbb{S}_4$.
- $(\mathbb{Z}_3)^2 \rtimes \mathbb{Z}_2$.
- $\left((\mathbb{Z}_3)^2 \rtimes \mathbb{Z}_2\right) \rtimes \mathbb{Z}_2$.

- $\mathbb{S}_5$.
- $(\mathbb{Z}_3)^3 \rtimes \mathbb{S}_4$.

Remark 2.58. *There was an earlier classification of such groups, due to B. Segre (see [33]). However, the group $\mathbb{Z}_8$ was missing from that classification, so a later result was presented here.*

Dimension-independent considerations

Proposition 2.59. *Let $G_0 \subset SL_N(\mathbb{C})$ be a finite subgroup, and $\bar{G}_0$ the image of its natural embedding into $PGL_N(\mathbb{C})$. Let $S \subset \mathbb{P}^{N-1}$ be a subvariety that is not contained in the union of any two proper linear subspaces of $\mathbb{P}^{N-1}$, and $\bar{G}_0$ fixes S point-wise. Then $\bar{G}_0$ is trivial.*

Proof: Pick $g \in G_0$. Then $<g>$ is a finite abelian group, and so (in some basis for $\mathbb{C}^N$) consists of diagonal matrices. Let $\bar{g}$ be the image of g under the natural projection into $\mathrm{PGL}_N(\mathbb{C})$.

Let $\bar{e}_1, \ldots, \bar{e}_N \in S$ be distinct points, such that their lifts $e_1, \ldots, e_N$ to $\mathbb{C}^N$ span all of $\mathbb{C}^N$ (these exist, since S is not contained in a proper linear subspace of $\mathbb{P}^{N-1}$). Then e_i are eigenvectors of g, and let λ_i be the corresponding eigenvalues. Reordering e_i-s if necessary, let $1 \leq m \in \mathbb{Z}$ be such that $\lambda_1 = \ldots = \lambda_m$ and $\lambda_n \neq \lambda_m \; \forall m < n \leq N$.

Assume $m < N$. Then take $A \subset \mathbb{C}^N$ to be the linear subspace spanned by $e_1, \ldots, e_m$ and $B \subset \mathbb{C}^N$ to be the linear subspace spanned by $e_{m+1}, \ldots, e_N$. Let $\bar{A}, \bar{B}$ be their natural projections into $\mathbb{P}^{N-1}$. These are proper linear subspaces, so $\exists \bar{p} \in S \setminus (\bar{A} \cup \bar{B})$.

This means there are at least $1 + N - m$ distinct linear eigenspaces for g not contained in A, at least one of which is not contained in B either. Therefore must have $\lambda_n = \lambda_m$ for some $m < n \leq N$, contradicting the choice of m.

This means $m = N$, and so g is a scalar matrix. □

Corollary 2.60. *Let $G \subset SL_N(\mathbb{C})$ and let $\bar{G}$ be its natural projection into $PGL_N(\mathbb{C})$. Let $S \subset \mathbb{P}^{N-1}$ be a $\bar{G}$-invariant subvariety, that is not contained in the union of any two proper linear subspaces of $\mathbb{P}^{N-1}$. Let $\pi_s : \bar{G} \to Aut(S)$ be the natural homomorphism. Then $ker(\pi_S) = \{Id_{\bar{G}}\}$.*

Remark 2.61. *If $S \subset \mathbb{P}^{N-1}$ is an irreducible surface, then either it is contained in a hyperplane of $\mathbb{P}^{N-1}$ or it is not contained in the union of any two proper linear subspaces of $\mathbb{P}^{N-1}$.*

Let $N \geq 2$, $G \subset \mathrm{SL}_N(\mathbb{C})$ be a finite irreducible subgroup, and $\bar{G}$ its natural projection into $\mathrm{PGL}_N(\mathbb{C})$. Let $S \subset \mathbb{P}^{N-1}$ be a $\bar{G}$-invariant subvariety.

Proposition 2.62. *The group $\bar{G}$ is not cyclic.*

Proof: In that case G is abelian (as it is a central extension of $\bar{G}$), and all irreducible representations of abelian groups are 1-dimensional. □

Proposition 2.63. *S cannot be contained in a proper linear subspace of $\mathbb{P}^{N-1}$.*

Proof: If it is, then $\bar{G}$ must fix the smallest such subset, which corresponds to a proper linear G-invariant subset of $\mathbb{C}^N$. This means G cannot be irreducible, contradiction the assumption that it is. $\square$

Lemma 2.64. *In the notation above, let S contain a $\bar{G}$-orbit consisting of k isolated points. Then either $k = 0$ or $k \geq N$.*

Proof: These points define a G-invariant set of k lines in $\mathbb{C}^N$. If $k \neq 0$, these lines span a non-empty subspace of $\mathbb{C}^N$ of dimension at most k. Since G is irreducible, this subset must have dimension N, making $k \geq N$. $\square$

Group actions on a smooth quadric surface in $\mathbb{P}^3$

In this section some notation for a specific type of group action on $\mathbb{P}^3$ will be presented. This is a general form for an action that preserves a smooth quadric surface in $\mathbb{P}^3$ (which can be taken to be $\mathbb{P}^1 \times \mathbb{P}^1$ embedded via the Segre embedding). Some of the notation was taken from this action's description in [14, Section 4.3], with some additions tailored to describing individual members of families of related groups.

It is possible to present $\mathbb{P}^3$ as the set of non-zero 2×2 matrices modulo the scalar ones. From now on, consider the "matrix form" of $\mathbb{P}^3$ to be:

$$(x : y : u : v) \rightsquigarrow \begin{pmatrix} x & y \\ u & v \end{pmatrix}$$

Let S be the image of $\mathbb{P}^1 \times \mathbb{P}^1$ under the Segre embedding

$$((a : b), (c : d)) \in \mathbb{P}^1 \times \mathbb{P}^1 \mapsto (ac : ad : bc : bd) \in \mathbb{P}^3$$

Then S is the zero set of the determinant of the matrix form of $\mathbb{P}^3$.

Let $\bar{G}$ be a finite group acting faithfully on $\mathbb{P}^1 \times \mathbb{P}^1$. This variety has exactly two rulings, which $\bar{G}$ can either preserve or interchange. Consider the exact sequence

$$1 \longrightarrow H \longrightarrow \bar{G} \longrightarrow \mathbb{S}_2$$

where the image of $\bar{G}$ in $\mathbb{S}_2$ shows how $\bar{G}$ permutes the two rulings. Then $H \trianglelefteq \bar{G}$ is the maximal subgroup that preserves the ruling, and either $\bar{G} = H$ or $\bar{G}$ is generated by H and an element $\sigma\Omega$, where Ω is the involution interchanging the two rulings, and σ is some automorphism of S preserving the ruling, with $\sigma^2 \in H$ (see [14, Theorem 4.9]).

Let $\pi_i : H \to H_i$ be the projections of H on the automorphism groups of the two components of $\mathbb{P}^1 \times \mathbb{P}^1$. Then have two more short exact sequences

$$1 \longrightarrow K_2 \longrightarrow H \longrightarrow H_1 \longrightarrow 1$$

$$1 \longrightarrow K_1 \longrightarrow H \longrightarrow H_2 \longrightarrow 1$$

It is clear that $K_1 \cap K_2 = \{1\}$. Therefore, for $i \neq j$ $(i, j \in \{1, 2\})$,

$$K_i \cong \hat{K}_i := \{kK_j \mid k \in K_i\} \trianglelefteq H/K_j \cong H_i$$

In this notation, $H_1/\hat{K}_1 \cong H/(K_1K_2) \cong H_2/\hat{K}_2$, so the group can be defined completely by $(H_1, K_1, H_2, K_2)_\alpha$, where α is an isomorphism $H_1/\hat{K}_1 \to H_2/\hat{K}_2$. In return, if H is known, one can reconstruct α by making it map

$$\pi_1(h)\hat{K}_1 \mapsto \pi_2(h)\hat{K}_2 \ (\forall h \in H).$$

In the matrix form described above, H_1 acts on $\mathbb{P}^3$ by left matrix multiplication, and H_2 acts by transposed right matrix multiplication. The involution switching the two rulings of $\mathbb{P}^1 \times \mathbb{P}^1$ corresponds to transposing the matrix. If $h_1 \in H_1$ acts on the first component of $\mathbb{P}^1 \times \mathbb{P}^1$ and $h_2 \in H_2$ acts on the second one, then write $\langle h_1 \,|\, h_2\rangle$ to denote this action. Explicitly, for any 2×2 matrices A and B,

$$\langle A \,|\, B\rangle \left(\begin{pmatrix} x & y \\ u & v \end{pmatrix}\right) = A \begin{pmatrix} x & y \\ u & v \end{pmatrix} B^T$$

Furthermore, interchanging the rulings of $\mathbb{P}^1 \times \mathbb{P}^1$ corresponds to transposing the matrix form of $\mathbb{P}^3$. It is worth noting that if $\bar{G} \neq H$, then conjugation by $\sigma\Omega$ provides isomorphisms $H_1 \cong H_2$ and $K_1 \cong K_2$.

Note that given any matrix presentations of the lifts of H_1 and H_2 to $\mathrm{SL}_2(\mathbb{C})$, it is possible to choose a basis for $\mathbb{P}^3$, such that the matrices this action of G uses are exactly the corresponding elements of H_1 and H_2 in the given presentation. Unless specified otherwise, assume throughout that the basis has been chosen to correspond to the presentations in 2.17.

The following notation may sound somewhat unnecessary, but it will assist in avoiding describing explicit generators of groups later on. For some groups, one can say some of its elements are of some special "type", e.g. the order 2 element in $\mathbb{D}_{2n}$ (n odd) or the 3-cycles in $\mathbb{A}_4$. Given a finite subgroup $(H_1, K_1, H_2, K_2)_\alpha$ with $H_1, H_2 \subset \mathrm{SL}_2(\mathbb{C})$, say elements of H_1 and H_2 of some fixed type are "coupled", if $\forall \langle h \,|\, h'\rangle \in H$ with h an element of this type, then h' must be either an element of the same type or a product of such an element and an element of a different type. Otherwise say that elements of this type are "not coupled". For example, if $H_1 \cong H_2 \cong \mathbb{D}_{10}$, then order 2 elements b are not coupled in G if G contains an element $\langle a^k \,|\, b\rangle$ for some $k, l \in \mathbb{Z}$, where a is an order 5 element. Otherwise they are coupled.

Finally, all the possible isomorphism classes for the groups acting on the smooth quadric surface in $\mathbb{P}^3$ have been classified and are as follows:

Theorem 2.65 ([14, §4.3])**.** *Let G be a finite subgroup of $PGL_2(\mathbb{C}) \times PGL_2(\mathbb{C})$. Then one of the following holds:*

- *$G \cong A \times B$, where A and B are finite subgroups of $PGL_2(\mathbb{C})$. These are called the* product subgroups.
- *$G \cong \{(g, \alpha(g)) \in A \times A \mid g \in A\} \cong A$ where A is a finite subgroup $PGL_2(\mathbb{C})$, and $\alpha \in Aut(A)$. These are called the* twisted diagonal subgroups.
- *G is conjugate to one of the following groups or its image under the switching of factors:*
  - $\frac{1}{2}[\mathbb{S}_4 \times \mathbb{S}_4] \cong (\mathbb{A}_4 \times \mathbb{A}_4) \rtimes \mathbb{Z}_2$.

- $\frac{1}{6}[\mathbb{S}_4 \times \mathbb{S}_4] \cong (\mathbb{Z}_2)^4 \rtimes \mathbb{S}_3$.
- $\frac{1}{3}[\mathbb{A}_4 \times \mathbb{A}_4] \cong (\mathbb{Z}_2)^4 \rtimes \mathbb{Z}_3$.
- $\frac{1}{2}[\mathbb{D}_{2m} \times \mathbb{D}_{4n}] \cong (\mathbb{Z}_m \times \mathbb{D}_{2n}) \rtimes \mathbb{Z}_2$ *($m, n \geq 2$).*
- $\frac{1}{4}[\mathbb{D}_{4m} \times \mathbb{D}_{4n}] \cong (\mathbb{Z}_m \times \mathbb{Z}_n) \rtimes \mathbb{Z}_4$ *(m, n odd).*
- $\frac{1}{2k}[\mathbb{D}_{2mk} \times \mathbb{D}_{2nk}]_s \cong (\mathbb{Z}_m \times \mathbb{Z}_n) \rtimes \mathbb{D}_{2k}$ *($(s,k) = 1$).*
- $\frac{1}{2k}[\mathbb{D}_{2mk} \times \mathbb{D}_{2nk}]_s \cong (\mathbb{Z}_m \times \mathbb{Z}_n) \rtimes \mathbb{D}_{2k}$ *($(s,2k) = 1$; m, n odd).*
- $\frac{1}{k}[\mathbb{Z}_{mk} \times \mathbb{Z}_{nk}]_s \cong (\mathbb{Z}_m \times \mathbb{Z}_n) \rtimes \mathbb{Z}_k$ *($(s,k) = 1$).*
- $\frac{1}{k}[\mathbb{Z}_{mk} \times \mathbb{Z}_{nk}]_s \cong (\mathbb{Z}_m \times \mathbb{Z}_n) \rtimes \mathbb{Z}_k$ *($(s,2k) = 1$; m, n odd).*
- $\frac{1}{2}[\mathbb{D}_{2m} \times \mathbb{S}_4] \cong (\mathbb{Z}_m \times \mathbb{A}_4) \rtimes \mathbb{Z}_2$.
- $\frac{1}{2}[\mathbb{D}_{4m} \times \mathbb{S}_4] \cong (\mathbb{D}_{2m} \times \mathbb{A}_4) \rtimes \mathbb{Z}_2$ *($m \geq 2$).*
- $\frac{1}{6}[\mathbb{D}_{6m} \times \mathbb{S}_4] \cong \left(\mathbb{Z}_m \times (\mathbb{Z}_2)^2\right) \rtimes \mathbb{S}_3$ *($m \geq 2$).*
- $\frac{1}{2}[\mathbb{Z}_{2m} \times \mathbb{S}_4] \cong (\mathbb{Z}_m \times \mathbb{A}_4) \rtimes \mathbb{Z}_2$ *($m \geq 2$).*
- $\frac{1}{3}[\mathbb{Z}_{3m} \times \mathbb{A}_4] \cong \left(\mathbb{Z}_m \times (\mathbb{Z}_2)^2\right) \rtimes \mathbb{Z}_3$ *($m \geq 2$).*
- $\frac{1}{2}[\mathbb{D}_{4m} \times \mathbb{D}_{4n}] \cong (\mathbb{D}_{2m} \times \mathbb{D}_{2n}) \rtimes \mathbb{Z}_2$ *($m, n \geq 2$).*
- $\frac{1}{2}[\mathbb{Z}_{2m} \times \mathbb{D}_{4n}] \cong (\mathbb{Z}_m \times \mathbb{D}_{2n}) \rtimes \mathbb{Z}_2$ *($n \geq 2$).*
- $\frac{1}{2}[\mathbb{Z}_{2m} \times \mathbb{D}_{2n}] \cong (\mathbb{Z}_m \times \mathbb{Z}_n) \rtimes \mathbb{Z}_2 \cong \mathbb{Z}_m \rtimes \mathbb{D}_{2n}$.

All finite subgroups of $Aut(\mathbb{P}^1 \times \mathbb{P}^1)$ are either conjugate to a finite subgroup of $PGL_2(\mathbb{C}) \times PGL_2(\mathbb{C})$ or conjugate to $G^0 \rtimes \mathbb{Z}_2$, where G^0 is a finite subgroup of $PGL_2(\mathbb{C}) \times PGL_2(\mathbb{C})$ and $\mathbb{Z}_2$ is generated by an automorphism which interchanges the rulings (denoted $\sigma\Omega$ above).

It is easy to see from the description, that all these automorphism groups can be represented using the notation described in this section.

Chapter 3

Low dimensional classification

In this chapter, the finite groups giving rise to low-dimensional (dimensions 3, 4 and 5) singularities that are not weakly exceptional will be classified. These results originally appeared in my papers [31] and [32], and the treatment here follows that in the original papers. Although parts of that proof can be streamlined by using the ideas introduced in Chapter 4, such optimisation has been omitted in order to show the different possible approaches to the problem, as well as to illustrate the ideas behind the approach used in Chapter 4.

3.1 Three-dimensional case

Let $G \subset \mathrm{SL}_3(\mathbb{C})$ be a finite irreducible subgroup. Recall that, by Theorem 2.48, the singularity induced by G is not weakly-exceptional if and only if G has a degree 2 semi-invariant. Also recall that the list of all possible groups G is known, given in Proposition 2.18.

The exceptional singularities in dimension 3 have been classified by:

Theorem 3.1 (see [25]). *The group G induces an exceptional singularity if and only if $\bar{G}$ is isomorphic to $\mathbb{A}_6$, Klein's simple group $\mathbb{K}_{168}$ of size 168, Hessian group $\mathbb{H}_{648}$ of size 648 or its normal subgroup F_{216} of size 216.*

Thus, it remains to decide which of the remaining groups induce weakly exceptional singularities (since any exceptional singularity is weakly exceptional by Lemma 2.33).

If a group had a semi-invariant of degree 1 (i.e. a G-invariant plane), this would mean that the group is not irreducible. Therefore, one can assume no such semi-invariant exists and concentrate on those of degree 2. This makes two possible approaches to this classification viable:

1. Since both $h^0\left(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2)\right) = 6$ and the number of possible families of isomorphism classes for G are small, it is possible to explicitly check all the possible semi-invariants of every family of groups. This is a perfectly viable approach in this low dimension, but it stops being effective (at least, in its naive form) even in dimension 4, due to a very rapid growth of both of these numbers.

2. The criteria for weak exceptionality (in any given dimension) can be stated in terms of non-existence of a certain list of $\bar{G}$-invariant subvarieties of $\mathbb{P}^{N-1}$. It is therefore possible to consider the automorphism groups of these varieties and to deduce that G cannot be a lifting of one of them to an action on $\mathbb{C}^N$. This is also a feasible approach, but it has the drawback of relying on a classification of suitable subvarieties and their automorphism groups.

In this section, the proof will be done by the second method, since it is more clear and concise. However, a small section of the proof via the first method will also be presented, as this method naturally evolves into one that is useful in higher dimensions.

The classification can be obtained as follows:

Lemma 3.2. *Assume that the singularity induced by G is not weakly-exceptional. Then:*

- *$\bar{G}$ leaves a smooth curve $C \subset \mathbb{P}^2$ of degree 2 invariant.*
- *G is isomorphic to a central extension of one of $\mathbb{D}_{2n}$ (some $n \geq 2$), $\mathbb{A}_4$, $\mathbb{S}_4$ or $\mathbb{A}_5$ by scalar elements.*

Proof: By Theorem 2.48, $\bar{G}$ must preserve a curve $C \subset \mathbb{P}^2$ of degree 2. If C is singular, then it must have exactly one isolated singularity. Then this singularity must be a $\bar{G}$-orbit of size 1, which is impossible by Lemma 2.64. Therefore, C must be smooth and hence rational, with $\bar{G}$ isomorphic to a finite subgroup of $\mathrm{Aut}\,(\mathbb{P}^1)$, as classified in Theorem 2.17. Since by Proposition 2.62, $\bar{G}$ cannot be cyclic, it must be isomorphic to one of $\mathbb{D}_{2n}$ (some $n \geq 2$), $\mathbb{A}_4$, $\mathbb{S}_4$ or $\mathbb{A}_5$. The group G must then be isomorphic to one of its (possibly, trivial) central extensions by scalar elements. □

Now it remains to identify the "suspect" groups in the classification in Proposition 2.18 and to check that they indeed have the necessary semi-invariants.

Theorem 3.3 (Main theorem in dimension 3)**.** *Let $G \subset SL_3(\mathbb{C})$ be a finite subgroup. Then G induces a weakly-exceptional but not an exceptional singularity if and only if one of the following holds:*

- *G is an irreducible monomial group, and $\bar{G}$ is not isomorphic to $(\mathbb{Z}_2)^2 \rtimes \mathbb{Z}_3$ or $(\mathbb{Z}_2)^2 \rtimes \mathbb{S}_3$.*
- *G is isomorphic to the group E_{108} of size 108.*

Proof: Compare the lists of groups in Proposition 2.18 and Lemma 3.2. It is easy to see that:

- If $\bar{G} \cong \mathbb{D}_{2n}$, then G must be a group of type 2.
- If $\bar{G} \cong \mathbb{A}_4$, then G must be a central extension of $(\mathbb{Z}_2)^2 \rtimes \mathbb{Z}_3 \cong \mathbb{A}_4$, where (in some basis) both $\mathbb{Z}_2$-s act diagonally and $\mathbb{Z}_3$ permutes the basis. Such a group is of type (3).

- If $\bar{G} \cong \mathbb{S}_4$, then G must be a central extension of $(\mathbb{Z}_2)^2 \rtimes \mathbb{S}_3 \cong \mathbb{S}_4$, where (in some basis) both $\mathbb{Z}_2$-s act diagonally and $\mathbb{S}_3$ permutes the basis. Such a group is of type (4).

- If $\bar{G} \cong \mathbb{A}_5$, then G, as its central extension, must be of type (8) or (10)

The groups of types (1) and (2) are not irreducible, and thus the singularities they induce are not weakly exceptional. Consider the central extensions of $\mathbb{A}_4$ and $\mathbb{S}_4$. Use their presentations from Proposition 2.18, and let (x, y, z) be the corresponding coordinates for $\mathbb{C}^3$. Then the groups have a semi-invariant smooth conic defined by $x^2 + y^2 + z^2 = 0$ and thus the singularities they induce are not weakly exceptional either. Similarly, given any presentation of a group of type (8) or (10), one can easily find a smooth conic curve the group preserves (see, for example, [42, Section 2.9]).

Since it was shown in Lemma 3.2 that the singularities all other groups induce must be weakly exceptional, Theorem 3.3 follows by excluding these groups as well as the groups inducing exceptional singularities (as listed in Theorem 3.1) from the list of all irreducible finite subgroups of $\mathrm{SL}_3(\mathbb{C})$. $\square$

As stated at the start of this section, here is a fragment of the proof of the same result via the other method:

Example 3.4. *The singularity induced by the group*

$$G = E_{108} \subset F_{216} \subset \mathbb{H}_{648} \subset SL_3(\mathbb{C})$$

is weakly exceptional.

Proof: Use the group presentation and notation from Proposition 2.18 throughout. According to Theorem 2.48, it is enough to prove that the group has no semi-invariants of degree 1 or 2.

Assume the group has a semi-invariant $f_1(x, y, z)$ of degree 1. Then

$$f_1(x, y, z) = \alpha x + \beta y + \gamma z$$

for some $\alpha, \beta, \gamma \in \mathbb{C}$, not all zero. By the linearity of the action of G, can assume $\alpha \in \{0, 1\}$. Since f_1 is G-semi-invariant, $T(f_1) = \mu_T f_1$ for some $\mu_T \in \mathbb{C}$. Since $T^3 = \mathrm{I}_3$, $\mu_T^3 = 1$ and $\mu_T \neq 0$. Thus,

$$\alpha y + \beta z + \gamma x = \mu_T (\alpha x + \beta y + \gamma z)$$

This implies that $\alpha = 1$, $\beta = \mu_T^2$ and $\gamma = \mu_T$, i.e.

$$f_1(x, y, z) = x + \mu_T^2 y + \mu_T z$$

Similarly, $S(f_1) = \mu_S f_1$ for some $\mu_S \in \mathbb{C}$. Since $S^3 = \mathrm{I}_3$, $\mu_S^3 = 1$ and $\mu_S \neq 0$. Thus,

$$\mu_S \left(x + \mu_T^2 y + \mu_T z\right) = x + \mu_T^2 \omega y + \mu_T \omega^2 z$$

This immediately implies that $\mu_S = 1$, which is impossible, since $\omega \neq 1$. Therefore, no such f_1 exists.

Now assume the group has a semi-invariant $f_2(x, y, z)$ of degree 2. Then

$$f_2(x, y, z) = \alpha x^2 + \beta y^2 + \gamma z^2 + \delta xy + \varepsilon xz + \eta yz$$

for some $\alpha, \beta, \gamma, \delta, \varepsilon, \eta \in \mathbb{C}$, not all zero.

Applying the matrix T, get $T(f_2) = \lambda_T f_2$ for some $\lambda_T \in \mathbb{C}$. Since $T^3 = \mathrm{I}_3$, $\lambda_T^3 = 1$ and $\lambda_T \neq 0$. Thus,

$$\alpha y^2 + \beta z^2 + \gamma x^2 + \delta yz + \varepsilon xy + \eta xz = \lambda_T \left(\alpha x^2 + \beta y^2 + \gamma z^2 + \delta xy + \varepsilon xz + \eta yz\right)$$

This implies that $\beta = \lambda_T^2 \alpha$, $\gamma = \lambda_T \alpha$, $\varepsilon = \lambda_T \delta$ and $\eta = \lambda_T^2 \delta$, i.e.

$$f_2(x, y, z) = \alpha \left(x^2 + \lambda_T^2 y^2 + \lambda_T z^2\right) + \delta \left(xy + \lambda_T xz + \lambda_T^2 yz\right)$$

with α and δ not both zero.

Applying the matrix S, get $S(f_2) = \lambda_S f_2$ for some $\lambda_S \in \mathbb{C}$. Since $S^3 = \mathrm{I}_3$, $\lambda_S^3 = 1$ and $\lambda_S \neq 0$. Thus,

$$\begin{aligned} \alpha \left(x^2 + \lambda_T^2 \omega^2 y^2 + \lambda_T \omega z^2\right) + \delta \left(\omega xy + \lambda_T \omega^2 xz + \lambda_T^2 yz\right) &= \\ = \ \lambda_S \left(\alpha \left(x^2 + \lambda_T^2 y^2 + \lambda_T z^2\right) + \delta \left(xy + \lambda_T xz + \lambda_T^2 yz\right)\right) \end{aligned}$$

This implies that either $\lambda_S = 1$ or $\alpha = \delta = 0$. Since α and δ cannot both be zero, deduce that $\lambda_S = 1$. But this is impossible, since that would imply that $\omega = 1$, which it is not. Therefore, no such f_2 exists.

Thus the group does not have any semi-invariants of degree 1 or 2, and therefore induces a weakly exceptional singularity. □

Remark 3.5. *Clearly, this method is not as "clean" as the previous one. However, it has the advantage of not needing the classification of any varieties to work, and will therefore be "more applicable" to situations in higher dimensions. The main idea to be observed here is that the calculations get greatly simplified once some monomials are shown to be in the same G-orbit (in this case, via the action of T). This idea will be relied upon heavily when considering the 5-dimensional case in Section 3.3 and the general prime-dimensional case in Chapter 4.*

3.2 Four-dimensional case

As mentioned above, the list of all possible subgroups of $\mathrm{SL}_4(\mathbb{C})$ is too long to be useful. Therefore, the list of subgroups inducing weakly exceptional singularities will have a similar drawback. Therefore, it makes sense to instead classify the finite subgroups of $\mathrm{SL}_4(\mathbb{C})$ that induce singularities that are *not* weakly exceptional. Furthermore, it is known from Theorem 2.45 that all the non-irreducible groups are on this list, so it makes sense to restrict attention to the irreducible ones. So, for this section, let $G \subset \mathrm{SL}_4(\mathbb{C})$ be a finite irreducible subgroup, and $\bar{G}$ its image under the natural projection $\mathrm{SL}_4(\mathbb{C}) \to \mathrm{PGL}_4(\mathbb{C})$.

In view of Theorem 2.49, this means that $\bar{G}$ must be a finite subgroup of an automorphism group of a surface of degree 2 or 3, or of a smooth rational cubic curve in $\mathbb{P}^3$. First consider the rational curve case.

Proposition 3.6. *Assume that there exists a $\bar{G}$-invariant smooth rational cubic curve in $\mathbb{P}^3$, but no $\bar{G}$-invariant surfaces of degree 2 or 3. Then $\bar{G} \cong \mathbb{A}_5$, and G is its central extension $\overline{\mathbb{A}}_5$ acting as the third symmetric power of its irreducible 2-dimensional representation.*

Proof: Let C be a $\bar{G}$-invariant smooth rational cubic curve. Then one can assume C is the image of

$$\mathcal{C} : (x : y) \in \mathbb{P}^1 \mapsto \left(x^3 : x^2 y : xy^2 : y^3\right) \in \mathbb{P}^3$$

This implies (by Proposition 2.60) that $\bar{G}$ must be isomorphic to one of the finite automorphism groups of $\mathbb{P}^1$, with its action induced by the action of $\mathbb{P}^1$ via $\mathcal{C}$. This means (by Theorem 2.17) that $\bar{G}$ must be one of the following:

- Cyclic group.
- Dihedral group.
- $\mathbb{A}_4$ or $\mathbb{S}_4$.
- $\mathbb{A}_5$.

No cyclic groups or their central extensions have any irreducible 4-dimensional representations. Therefore, $\bar{G}$ cannot be cyclic. The same arguments excludes the possibility of $\bar{G}$ being dihedral.

The induced actions of $\mathbb{A}_4$ and $\mathbb{S}_4$ preserve $\{(x : y : u : v) \in \mathbb{P}^3 \mid xy - uv = 0\}$, which is a smooth quadric surface. Therefore, $\bar{G}$ cannot be one of them.

Therefore, $\bar{G}$ cannot be isomorphic to anything except $\mathbb{A}_5$, with G acting as described above. Such an action is primitive, and so irreducible. It can be checked directly (or seen in, for example, [7, Proof of Lemma 4.9]) that this action does not preserve any projective surfaces of degree 2 or 3. □

It is now necessary to process the automorphism groups of the surfaces of degree 2 and 3. The first step would be to find out exactly what these automorphism groups might be isomorphic to. This can be achieved by:

Lemma 3.7. *Let $G \subset SL_4(\mathbb{C})$ be a finite irreducible group, $\bar{G} \subset PGL_4(\mathbb{C})$ its projection, and let $S \subset \mathbb{P}^3$ be a $\bar{G}$-invariant surface of degree minimal among the degrees of all $\bar{G}$-invariant surfaces. Then either $\deg S \geq 4$ or S is smooth.*

Proof: Since G is irreducible, $\deg S \geq 2$. If $\deg S = 2$ and S is singular, then either S has exactly one isolated singularity (which has to be a $\bar{G}$-fixed point, impossible by Lemma 2.64), or S is a union of two planes and thus has a singular line (impossible, since G is irreducible), which must then be $\bar{G}$-invariant. Therefore, if $\deg S = 2$ then S must be smooth.

If $\deg S = 3$, and S is not irreducible, then either it is the union of a plane and an irreducible quadric surface (each of which must thus be a $\bar{G}$-invariant surface of smaller degree, contradicting the minimality of the degree of S) or S is the union of 3 distinct planes, whose intersection gives either a point or a line fixed by all of $\bar{G}$ (stopping G from being irreducible). Hence S is irreducible.

Assume that $\deg S = 3$ and S has non-isolated singularities, with C being the union of all singular curves on S. Then, one can easily see that C is a line. Since $\bar{G}(S) = S$, must have $\bar{G}(C) = C$, and so there exists a $\bar{G}$-invariant line, contradicting irreducibility of G. Therefore if $\deg S = 3$ then S must have at worst isolated singularities.

If $\deg S = 3$ and S is singular with only isolated singularities, then by [3], the singularity types form one of the following collections: (A_1), $(2A_1)$, (A_1, A_2), $(3A_1)$, (A_1, A_3), $(2A_1, A_2)$, $(4A_1)$, (A_1, A_4), $(2A_1, A_3)$, $(A_1, 2A_2)$, (A_1, A_5). Given any type of singularity, the set of such singularities on S must be preserved by the action of $\bar{G}$. Therefore, by Lemma 2.64, it must either be empty or have size at least 4. Therefore, S has to have exactly four A_1 singularities. Since there is only one such surface (see, for example, [3, proof of Lemma 3]), S must be the Cayley cubic, defined (in some basis) by

$$S = \left\{(x : y : u : v) \in \mathbb{P}^3 \mid xyu + xyv + xuv + yuv = 0\right\}$$

This surface, contains exactly 9 lines, six of which pass through pairs of singular points and the other three are defined by

$$\begin{aligned} x + y = 0 = u + v \\ x + u = 0 = y + v \\ x + v = 0 = y + u \end{aligned}$$

These last three lines must therefore be mapped to each other by all of $\bar{G}$. But since they are coplanar, $\bar{G}$ preserves the plane they lie in, contradicting the irreducibility assumption for G. Thus if S is a cubic surface, then it must be smooth. $\square$

It now remains to consider exactly two cases:

1. There exists a $\bar{G}$-invariant smooth quadric surface.
2. There are no $\bar{G}$-invariant quadric surfaces, but there is a $\bar{G}$-invariant smooth cubic surface.

The two cases will be considered separately below.

Lemma 3.8. *If $G \subset SL_4(\mathbb{C})$ is a finite irreducible subgroup, and $\bar{G}$ its projection to $PGL_4(\mathbb{C})$. Also assume that there is no $\bar{G}$-invariant quadric surface, and $S \subset \mathbb{P}^3$ is a smooth $\bar{G}$-invariant cubic surface. Then G must be isomorphic to a central extension of one of:*

- $\left((\mathbb{Z}_3)^3 \rtimes \mathbb{Z}_2\right) \rtimes \mathbb{Z}_2$.
- $(\mathbb{Z}_3)^3 \rtimes \mathbb{S}_4$.

by scalar elements, acting as described below. Both these cases produce monomial actions.

Proof: Since $\bar{G} \subset \mathrm{Aut}(S)$ is a finite subgroup, so by Theorem 2.57, $\bar{G}$ must be isomorphic to one of:

1. Cyclic groups $\{\mathrm{Id}_{\bar{G}}\}$, $\mathbb{Z}_2$, $\mathbb{Z}_4$, $\mathbb{Z}_8$.

2. Dihedral groups $(\mathbb{Z}_2)^2 \cong \mathbb{D}_4$, $\mathbb{S}_3 \cong \mathbb{D}_6$, $\mathbb{S}_3 \times \mathbb{Z}_2 \cong \mathbb{D}_{12}$.

3. $\mathbb{S}_4$.

4. $(\mathbb{Z}_3)^2 \rtimes \mathbb{Z}_2$ or $\left((\mathbb{Z}_3)^3 \rtimes \mathbb{Z}_2\right) \rtimes \mathbb{Z}_2$.

5. $\mathbb{S}_5$.

6. $(\mathbb{Z}_3)^3 \rtimes \mathbb{S}_4$.

These cases will be considered separately:

1. The group $\bar{G}$ cannot be cyclic by Proposition 2.62.

2. Dihedral groups and their extensions by scalar elements do not have any irreducible 4-dimensional representations, so these groups cannot act irreducibly.

3. The group $\mathbb{S}_4$ by itself has no 4-dimensional irreducible representations, but its central extension has (up to a choice of a root of unity) only one such. This representation preserves a quadric surface (see the twisted diagonal actions in Lemma 3.12).

4. For convenience, write $\bar{G}' = (\mathbb{Z}_3)^2 \rtimes \mathbb{Z}_2$ and $\bar{G}'' = \left((\mathbb{Z}_3)^3 \rtimes \mathbb{Z}_2\right) \rtimes \mathbb{Z}_2$, with all the notation following in the obvious manner (i.e. write G' for the lift of $\bar{G}'$ to $\mathrm{SL}_4(\mathbb{C})$, etc.).

   Using the notation from [20], define the group G^9_{54} generated by elements $\bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\delta}$ with $\bar{\alpha}^3 = \bar{\beta}^3 = \bar{\gamma}^3 = \bar{\delta}^2 = \mathrm{id}$, $\bar{\alpha}$ generating the centre of G^9_{54} and $G^9_{54}/C\left(G^9_{54}\right) \cong (\mathbb{Z}_3)^2 \rtimes \mathbb{Z}_2$.

   Then one can see that $\bar{G}' = G^9_{54}/C\left(G^9_{54}\right)$ and $\bar{G}'' = G^9_{54} \rtimes \mathbb{Z}_2$ (with additional generator $\bar{\epsilon}$, such that $\bar{\epsilon}^2 = \mathrm{id}$). Let $\alpha, \beta, \gamma, \delta, \epsilon$ be lifts of $\bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\delta}$ (respectively) to $\mathrm{SL}_4(\mathbb{C})$.

   Let $h_1 := \alpha^3$, $h_2 := \beta^3$. $\bar{\alpha}, \bar{\beta}$ commute, so say $\beta\alpha = \alpha\beta h_3$. By the structure of the lift, h_i are scalar matrices of order 1, 2 or 4. Then

   $$h_1^3 h_2 = \left(\alpha^2\beta\alpha\right)^3 = (\beta h_1 h_3)^3 = h_2 h_1^3 h_3^3$$

   and so $h_3 = \mathrm{id}$. Similarly, get α, β, γ all commuting. Hence the corresponding matrices can all be taken to be diagonal (by choosing a suitable basis). It is then easy to see that δ and ϵ must act as elements of a central extension of $\mathbb{S}_4$ permuting the basis.

   Since $\bar{G}''$ has only one normal subgroup of index 2, and $\bar{G}''$ has no centre (otherwise $\bar{G}''/C\left(\bar{G}''\right)$ would be on the list of groups acting on a cubic surface), $\bar{\delta}\bar{\epsilon} \neq \bar{\epsilon}\bar{\delta}$. Therefore, up to conjugation, δ interchanges the first and the second basis vectors, and ϵ interchanges the first basis vector with the third one and the second basis vector with the fourth one.

This means that G' is not irreducible, while G'' is irreducible, monomial and (up to conjugation) is generated by

$$\begin{pmatrix} \zeta_3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \zeta_3^{-1} \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \zeta_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \zeta_3^{-1} \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \zeta_3 & 0 \\ 0 & 0 & 0 & \zeta_3^{-1} \end{pmatrix},$$

$$\zeta_8 \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

This group leaves (for example) the cubic polynomial $x^3 + y^3 + z^3 + w^3$ (in coordinates (x, y, z, w) for $\mathbb{C}^4$) semi-invariant, and by direct computation, one sees that the group does not have a semi-invariant quadric surface.

5. The group $\mathbb{S}_5$ is, according to [33, §100], the automorphism group of the irreducible diagonal cubic surface

$$S = \left\{ (x_0 : x_1 : x_2 : x_3 : x_4) \in \mathbb{P}^4 \;\middle|\; \begin{array}{c} x_0^3 + x_1^3 + x_2^3 + x_3^3 + x_4^3 = 0, \\ x_0 + x_1 + x_2 + x_3 + x_4 = 0 \end{array} \right\}$$

which immediately implies that there exists a $\bar{G}$-invariant quadric surface. For the group's action on it, see Lemma 3.12.

6. As stated in [33, §100], the monomial group $(\mathbb{Z}_3)^3 \rtimes \mathbb{S}_4$ acts by permuting the basis vectors of $\mathbb{C}^4$ arbitrarily and multiplying them by arbitrary cube roots of unity. Hence (up to conjugation) G is a central extension of such a group by scalar elements.

   This group clearly leaves the cubic polynomial $x^3 + y^3 + z^3 + w^3$ (in coordinates (x, y, z, w)) semi-invariant, and by direct computation, one sees that the group does not have a semi-invariant quadric surface.

□

Quadric surface case

Now assume that there exists a $\bar{G}$-invariant smooth quadric surface $S \subset \mathbb{P}^3$. The rest of this section in devoted to creating the list of possible values of G (or, equivalently, $\bar{G}$) that can appear in this situation. The final list can be found in Lemma 3.12.

Since $S \subset \mathbb{P}^3$, there exists a basis for $\mathbb{P}^3$, in which S is the image of the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ into $\mathbb{P}^3$, as discussed in Section 2.4. The image of this embedding is

$$S = \left\{ (x : y : u : v) \in \mathbb{P}^3 \mid xv - yu = 0 \right\}$$

This implies that the subgroup of $\mathrm{PGL}_4(\mathbb{C})$ preserving S is isomorphic to the group $(\mathrm{PGL}_2(\mathbb{C}))^2 \rtimes \mathbb{Z}_2$, with the action that can be described in the notation given in Section 2.4. This notation will be used throughout the rest of this section.

The list of possible isomorphism classes for $\bar{G}$ can be found in Theorem 2.65. However, to be able to use the list, one needs to understand the type of action of the possible choices for $\bar{G}$, in other words, one needs to split the possibilities for G into the following categories:

- Not irreducible.
- Irreducible monomial.
- Irreducible non-monomial imprimitive.
- Primitive (hence irreducible).

This can be done directly by looking at the representations used to build the action. To make the explanations more simple, it will be assumed that H_1 and H_2 (see Section 2.4) both contain a non-scalar diagonal matrix. This will mean that any proper invariant subspace must have a basis that is also a subset of the chosen basis for $\mathbb{C}^4$. It is easy to check that for all the actions used in this section, there exists a basis for $\mathbb{C}^4$ in which the action contains such a matrix. Now fix this basis for the remainder of this section.

The first step would be to understand the conditions necessary for the irreducibility of G. This depends on the choices of H_1 and H_2 and on their interplay inside $\bar{G}$. Since by construction, H_1 and H_2 are finite subgroups of $\mathrm{PGL}_2(\mathbb{C})$, they both must be among those given in Theorem 2.17. Furthermore, if an element of $\bar{G}$ interchanges the two rulings of S, H_1 and H_2 must be conjugate (by that element) and therefore isomorphic.

Proposition 3.9. *If either H_1 or H_2 is cyclic, then G is not irreducible.*

Proof: Assume without loss of generality that H_1 is cyclic. Then the action of H_1 on $\mathbb{P}^1$ has a fixed point p. Without loss of generality, assume $p = (1:0) \in \mathbb{P}^1$, and so the action of $\bar{G}$ on S fixes $\{(1:0)\} \times \mathbb{P}^1 \subset \mathbb{P}^1 \times \mathbb{P}^1 = S$. Under the Segre embedding, this corresponds to the subvariety

$$\{u = v = 0\} \subset \left\{(x:y:u:v) \in \mathbb{P}^3\right\}$$

Therefore, this subvariety must be fixed under the action of $\bar{G}$, and so G is not irreducible. $\square$

Proposition 3.10. *If H_1 and H_2 are not cyclic, and if at least one of them is not dihedral, then G is irreducible.*

Proof: Without loss of generality, assume H_1 is not dihedral. Then, it must be true that $H_1 \in \{\mathbb{A}_4, \mathbb{S}_4, \mathbb{A}_5\}$. Assume that $V \subset \mathbb{C}^4$ is a G-invariant subspace. Recall that the basis has been chosen for the representations of H_1 and H_2 to correspond to those in Theorem 2.17. Let $p \in V$.

If $H_1 \in \{\mathbb{A}_4, \mathbb{S}_4\}$, then let h_1 be an element of order 3. If $H_1 = \mathbb{A}_5$, let h_1 be an even element of order 2. Take an element $g \in G$, that maps to $h_1 \in H_1$. Then $g(p)$ is not a multiple of p (as G is not scalar), and so the dimension of V is at least 2. Therefore, if G is not irreducible, there must exist two 2-dimensional G-invariant blocks V_1 and V_2, such that they together span all of $\mathbb{C}^4$, and G acts on each of them irreducibly.

Now pick $h_2 \in H_2$, that is not presented by a diagonal matrix, and let $g_2 \in G$ be a preimage of h_2. Pick a point $p \in V_1$. Then, due to the action of H_2 on $\mathbb{P}^1 \times \mathbb{P}^1 = S$, and the embedding of S into $\mathbb{P}^3$, $g_2(p) \notin V_1$. Therefore, V_1 cannot be G-invariant, and therefore, G must be irreducible. $\square$

Proposition 3.11. *Assume the groups H_1 and H_2 are dihedral. Set their standard generators to be:*

$$H_1 = \mathbb{D}_{2n} = <a_1, b_1 \mid a_1^n = b_1^2 = Id_{H_1}>,$$

$$H_2 = \mathbb{D}_{2m} = <a_2, b_2 \mid a_2^m = b_2^2 = Id_{H_2}> .$$

Then G is irreducible unless the elements of H_1 containing b_1 and those of H_2 containing b_2 are coupled in $\bar{G}$. The action of such G on $\mathbb{C}^4$ is monomial.

Proof: Consider the action of H_1 on $\mathbb{P}^1$. It has an orbit of exactly two points, call them $q_{1,1}$ and $q_{1,2}$, such that a_1 fixes both of them and $b_1(q_{1,1}) = q_{1,2}$. Similarly, pick $q_{2,1}, q_{2,2} \in \mathbb{P}^1$, such that a_2 fixes both of them and $b_2(q_{2,1}) = q_{2,2}$. Let $\bar{p}_1, \bar{p}_2, \bar{p}_3, \bar{p}_4 \in \mathbb{P}^3$ be the images of the elements $(q_{1,1}, q_{2,1})$, $(q_{1,1}, q_{2,2})$, $(q_{1,2}, q_{2,1})$ and $(q_{1,2}, q_{2,2})$ of $\mathbb{P}^1 \times \mathbb{P}^1 = S$ (respectively) under the embedding. Then each p_i corresponds to a line $V_i \subset \mathbb{C}^4$, and $\{V_1, V_2, V_3, V_4\}$ is a partition of imprimitivity of the action of G on $\mathbb{C}^4$. Therefore, the action of G is monomial.

Consider the way the action of G permutes the V_i-s. Let T be the corresponding permutation group. If the elements containing b_i are not coupled under the action of G, then it is easy to see that $T = \mathbb{S}_4$, and so G acts irreducibly. However, if the elements are coupled, then $T = <(1,4)(2,3)> \cong \mathbb{Z}_2 \subset \mathbb{S}_4$, and so the action of G is not irreducible. $\square$

The irreducibility discussion above can be summarised as Table 3.1.

Table 3.1: Irreducibility of G

		H_1: $\mathbb{A}_4$, $\mathbb{S}_4$, $\mathbb{A}_5$	H_1: $\mathbb{D}_{2m}$	H_1: $\mathbb{Z}_m$
H_2	$\mathbb{A}_4$, $\mathbb{S}_4$, $\mathbb{A}_5$	Irreducible	Irreducible	Not irreducible
H_2	$\mathbb{D}_{2m}$	Irreducible	Depends on action	Not irreducible
H_2	$\mathbb{Z}_n$	Not irreducible	Not irreducible	Not irreducible

Now assume that the action of G is irreducible. Then it is necessary to determine whether the action is primitive, monomial or imprimitive non-monomial.

By direct computation, it is easy to see that in most cases, the place of G in this classification depends on the matrices in H_i that have 3 or more non-zero entries, and how these matrices are combined in G, i.e. on the isomorphism $\alpha : H_1/\hat{K}_1 \to H_2/\hat{K}_2$ (as defined above). The only exception occurs when H_i are dihedral, G does interchange the ruling, but $\Omega \notin \bar{G}$ (hence $\sigma\Omega \in \bar{G}$ for some non-trivial σ of even degree) — in this case, the automorphism σ needs to be considered.

With this in mind, direct computation provides the following criteria (putting $i \neq j \in \{1, 2\}$):

- If H_1, H_2 dihedral and G irreducible, then G acts monomially.
- If $H_i \in \{\mathbb{A}_4, \mathbb{S}_4, \mathbb{A}_5\}$ and $H_j = \mathbb{D}_{2n}$, then the action of G is non-monomial imprimitive.
- If $H_1 \cong H_2 \cong \mathbb{A}_5$, then the action of G is primitive.
- If $H_1, H_2 \in \{\mathbb{A}_4, \mathbb{S}_4\}$ and the 3-cycles are not coupled, then the action of G is primitive.
- If $H_1 \cong H_2 \cong \mathbb{A}_4$ and the 3-cycles are coupled, then the action of G is monomial.
- If $H_i \cong \mathbb{S}_4$, $H_j \cong \mathbb{A}_4$ and the 3-cycles are coupled, then the action is imprimitive non-monomial.
- If $H_i \cong \mathbb{S}_4$, $H_j \cong \mathbb{S}_4$, the 3-cycles are coupled and the odd permutations are coupled, then the action is monomial.
- If $H_i \cong \mathbb{S}_4$, $H_j \cong \mathbb{S}_4$, the 3-cycles are coupled, but the odd permutations are not coupled, then the action is primitive.

This list is clearly not exhaustive, but it is sufficient for determining the nature of all the groups below.

Since $\bar{G}$ is a finite group leaving a smooth quadric S invariant, its action must be equal (as shown in Section 2.4) to a suitable action of one of the finite automorphism groups of a smooth 2-dimensional quadric. Thus $\bar{G}$ must be conjugate to the image of one of the finite groups given in 2.65.

In order to make the structure of each of the groups slightly more explicit, the group structure will also be given in the notation

$$(H_1, K_1, H_2, K_2)_\alpha ,$$

where H_i, K_i are as before, and α is the gluing isomorphism between $H_1/\hat{K}_1$ and $H_2/\hat{K}_2$. Where only one such isomorphism exists, α will be omitted. For each isomorphism class, several representations of the group can be chosen. However, it is clear that the different faithful representations will differ by at most an outer automorphism, so all the properties that are of interest in this discussion will be the same for all of them. Therefore, for each isomorphism class, any faithful representation of H_i can be chosen. For any group $(H_1, K_1, H_2, K_2)_\alpha$, there also exists a group $(H_2, K_2, H_1, K_1)_{\alpha^{-1}}$, which corresponds to the same group with the components of the ruling of the quadric swapped. These two groups are conjugate to each other.

Lemma 3.12. *If $\bar{G} \subset PGL_4(\mathbb{C})$ is a finite irreducible subgroup, G its lift to $SL_4(\mathbb{C})$ and $S \subset \mathbb{P}^3$ a smooth $\bar{G}$-invariant quadric surface, then $\bar{G}$ must be conjugate to one of the following groups:*

Groups leaving the ruling of $\mathbb{P}^1 \times \mathbb{P}^1$ invariant:

1. *Product subgroups $(H_1, H_1, H_2, H_2) \cong H_1 \times H_2$ for some finite subgroups $H_i \in Aut(\mathbb{P}^1)$. Taking different choices for H_1, H_2, get the following groups of the form $H_1 \times H_2$:*

   (a) *9 primitive groups when $H_1, H_2 \in \{\mathbb{A}_4, \mathbb{S}_4, \mathbb{A}_5\}$.*

   (b) *3 families of non-monomial imprimitive groups $\mathbb{D}_{2m} \times H_2$, for some $H_2 \in \{\mathbb{A}_4, \mathbb{S}_4, \mathbb{A}_5\}$.*

   (c) *1 family of monomial groups $\mathbb{D}_{2m} \times \mathbb{D}_{2n}$.*

2. *Twisted diagonal subgroups $(H_1, 1, H_1, 1)_\alpha \cong H_1$ for some finite subgroup $H_1 \in Aut(\mathbb{P}^1)$. This gives 3 families of groups, indexed by the choice of isomorphism α. They are:*

   (a) *Monomial groups isomorphic to $\mathbb{A}_4$ or $\mathbb{S}_4$.*

   (b) *Primitive groups isomorphic to $\mathbb{A}_5$.*

   The twisted diagonal groups isomorphic to the dihedral groups do not act irreducibly, as the relevant central extensions (which are either dihedral or dicyclic groups) do not have any 4-dimensional irreducible representations.

3. *$\frac{1}{2}[\mathbb{S}_4 \times \mathbb{S}_4] \cong (\mathbb{A}_4 \times \mathbb{A}_4) \rtimes \mathbb{Z}_2 \cong (\mathbb{S}_4, \mathbb{A}_4, \mathbb{S}_4, \mathbb{A}_4)$, a primitive group generated by elements corresponding to $\langle [12][34] \,|\, id \rangle$, $\langle id \,|\, [12][34] \rangle$, $\langle [123] \,|\, id \rangle$, $\langle id \,|\, [123] \rangle$ and $\langle (12) \,|\, (12) \rangle$.*

4. *$\frac{1}{2}[\mathbb{D}_{2m} \times \mathbb{S}_4] \cong (\mathbb{Z}_m \times \mathbb{A}_4) \rtimes \mathbb{Z}_2 \cong (\mathbb{D}_{2m}, \mathbb{Z}_m, \mathbb{S}_4, \mathbb{A}_4)$, a family of imprimitive non-monomial groups generated by $\langle a_m \,|\, id \rangle$, $\langle id \,|\, [12][34] \rangle$, $\langle id \,|\, [123] \rangle$ and $\langle b \,|\, (12) \rangle$.*

5. *$\frac{1}{2}[\mathbb{D}_{4m} \times \mathbb{S}_4] \cong (\mathbb{D}_{2m} \times \mathbb{A}_4) \rtimes \mathbb{Z}_2 \cong (\mathbb{D}_{4m}, \mathbb{D}_{2m}, \mathbb{S}_4, \mathbb{A}_4)$ $(m \geq 2)$, a family of imprimitive non-monomial groups generated by the action of $\langle a_{2m}^2 \,|\, id \rangle$, $\langle b \,|\, id \rangle$, $\langle id \,|\, [12][34] \rangle$, $\langle id \,|\, [123] \rangle$ and $\langle a_{2m}^m \,|\, (12) \rangle$.*

6. *$\frac{1}{6}[\mathbb{D}_{6m} \times \mathbb{S}_4] \cong (\mathbb{Z}_m \times \mathbb{V}_4) \rtimes \mathbb{S}_3 \cong (\mathbb{D}_{6m}, \mathbb{Z}_m, \mathbb{S}_4, \mathbb{V}_4)$, a family of imprimitive non-monomial groups generated by $\langle a_{3m}^3 \,|\, id \rangle$, $\langle id \,|\, [12][34] \rangle$, $\langle id \,|\, (13)(24) \rangle$, $\langle a_{3m}^m \,|\, [123] \rangle$ and $\langle b \,|\, (12) \rangle$.*

7. *$\frac{1}{6}[\mathbb{S}_4 \times \mathbb{S}_4] \cong (\mathbb{V}_4 \times \mathbb{V}_4) \rtimes \mathbb{S}_3 \cong (\mathbb{S}_4, \mathbb{V}_4, \mathbb{S}_4, \mathbb{V}_4)$, a monomial group generated by the elements $\langle [12][34] \,|\, id \rangle$, $\langle (13)(24) \,|\, id \rangle$, $\langle id \,|\, [12][34] \rangle$, $\langle id \,|\, (13)(24) \rangle$, $\langle [123] \,|\, [123] \rangle$ and $\langle (12) \,|\, (12) \rangle$.*

8. *$\frac{1}{3}[\mathbb{A}_4 \times \mathbb{A}_4] \cong (\mathbb{V}_4 \times \mathbb{V}_4) \rtimes \mathbb{Z}_3 \cong (\mathbb{A}_4, \mathbb{V}_4, \mathbb{A}_4, \mathbb{V}_4)$, a monomial group generated by the elements $\langle [12][34] \,|\, id \rangle$, $\langle (13)(24) \,|\, id \rangle$, $\langle id \,|\, [12][34] \rangle$, $\langle id \,|\, (13)(24) \rangle$ and $\langle [123] \,|\, [123] \rangle$.*

9. $\frac{1}{2}\left[\mathbb{D}_{2m} \times \mathbb{D}_{4n}\right] \cong (\mathbb{Z}_m \times \mathbb{D}_{2n}) \rtimes \mathbb{Z}_2 \cong (\mathbb{D}_{2m}, \mathbb{Z}_m, \mathbb{D}_{4n}, \mathbb{D}_{2n})$ *(for* $m, n \geq 2$*), a family of monomial groups generated by* $\langle a_m \,|\, id\rangle$, $\langle id \,|\, a_{2n}^2\rangle$, $\langle id \,|\, b\rangle$ *and* $\langle b \,|\, a_{2n}^n\rangle$.

10. $\frac{1}{4}\left[\mathbb{D}_{4m} \times \mathbb{D}_{4n}\right]_\alpha \cong (\mathbb{Z}_m \times \mathbb{Z}_n) \rtimes \mathbb{D}_4 \cong (\mathbb{D}_{4m}, \mathbb{Z}_m, \mathbb{D}_{4n}, \mathbb{Z}_n)_\alpha$ *(where* $\alpha(b) = a_{2n}^n$, $\alpha(a_{2m}^m) = b$*), a family of monomial groups generated by* $\langle a_{2m}^2 \,|\, id\rangle$, $\langle id \,|\, a_{2n}^2\rangle$, $\langle a_{2m}^m \,|\, b\rangle$ *and* $\langle b \,|\, a_{2n}^n\rangle$.

11. $\frac{1}{2}\left[\mathbb{D}_{4m} \times \mathbb{D}_{4n}\right] \cong (\mathbb{D}_{2m} \times \mathbb{D}_{2n}) \rtimes \mathbb{Z}_2 \cong (\mathbb{D}_{4m}, \mathbb{D}_{2m}, \mathbb{D}_{4n}, \mathbb{D}_{2n})$ *(for* $m, n \geq 2$*), a family of monomial groups generated by* $\langle a_{2m}^2 \,|\, id\rangle$, $\langle b \,|\, id\rangle$, $\langle id \,|\, a_{2n}^2\rangle$, $\langle id \,|\, b\rangle$ *and* $\langle a_{2m}^m \,|\, a_{2n}^n\rangle$.

Groups that interchange the rulings of $\mathbb{P}^1 \times \mathbb{P}^1$ *(via an element* $\sigma \circ \Omega$*):*

12. $(H_1 \times H_1) \rtimes \mathbb{Z}_2 \cong (H_1, H_1, H_1, H_1) \rtimes \mathbb{Z}_2$ *(*$H_1 \in Aut(\mathbb{P}^1)$*). Taking different choices for* H_1 *and bearing in mind that choosing* H_1 *to be* $\mathbb{Z}_n$ *produces a group that is not irreducible (see discussion above), get* 2 *families of monomial groups*

    (a) $(\mathbb{D}_{2n} \times \mathbb{D}_{2n}) \rtimes \mathbb{Z}_2$

    and 3 *families of primitive groups, all of them indexed by the possible involutions acting on* H_1*:*

    (b) $(\mathbb{A}_4 \times \mathbb{A}_4) \rtimes \mathbb{Z}_2$.
    (c) $(\mathbb{S}_4 \times \mathbb{S}_4) \rtimes \mathbb{Z}_2$.
    (d) $(\mathbb{A}_5 \times \mathbb{A}_5) \rtimes \mathbb{Z}_2$.

13. $H_1 \rtimes \mathbb{Z}_2 \cong (H_1, 1, H_1, 1)_\alpha$ *(*$H_1 \in Aut(\mathbb{P}^1)$*). This gives* 3 *families of groups, indexed by the choice of isomorphism* α*. They are:*

    (a) *Monomial groups isomorphic to* $\mathbb{D}_{4n} \rtimes \mathbb{Z}_2$.
    (b) *Monomial groups isomorphic to* $\mathbb{A}_4 \rtimes \mathbb{Z}_2$ *or* $\mathbb{S}_4 \rtimes \mathbb{Z}_2$.
    (c) *Primitive groups isomorphic to* $\mathbb{A}_5 \rtimes \mathbb{Z}_2$.

14. $((\mathbb{A}_4 \times \mathbb{A}_4) \rtimes \mathbb{Z}_2) \rtimes \mathbb{Z}_2 \cong (\mathbb{S}_4, \mathbb{A}_4, \mathbb{S}_4, \mathbb{A}_4) \rtimes \mathbb{Z}_2$*, a family of primitive groups.*

15. $((\mathbb{V}_4 \times \mathbb{V}_4) \rtimes \mathbb{S}_3) \rtimes \mathbb{Z}_2 \cong (\mathbb{S}_4, \mathbb{V}_4, \mathbb{S}_4, \mathbb{V}_4) \rtimes \mathbb{Z}_2$*, a family of monomial groups.*

16. $((\mathbb{V}_4 \times \mathbb{V}_4) \rtimes \mathbb{Z}_3) \rtimes \mathbb{Z}_2 \cong (\mathbb{A}_4, \mathbb{V}_4, \mathbb{A}_4, \mathbb{V}_4) \rtimes \mathbb{Z}_2$*, a family of monomial groups.*

17. $((\mathbb{Z}_m \times \mathbb{Z}_m) \rtimes \mathbb{D}_4) \rtimes \mathbb{Z}_2 \cong (\mathbb{D}_{4m}, \mathbb{Z}_m, \mathbb{D}_{4m}, \mathbb{Z}_m)_\alpha \rtimes \mathbb{Z}_2$ *(where* $\alpha(b) = a_{2m}^m$, $\alpha(a_{2m}^m) = b$*), a family of monomial groups.*

18. $((\mathbb{D}_{2m} \times \mathbb{D}_{2m}) \rtimes \mathbb{Z}_2) \rtimes \mathbb{Z}_2 \cong (\mathbb{D}_{4m}, \mathbb{D}_{2m}, \mathbb{D}_{4m}, \mathbb{D}_{2m}) \rtimes \mathbb{Z}_2$ *(*$m \geq 2$*), a family of monomial groups.*

From the way the groups are presented, it is easy to see that all the groups do need to be in the list.

Proof: Immediate from Theorem 2.65 and the discussion above. Irreducibility of these groups can be seen via Table 3.1. □

Theorem 3.13 (Main theorem in dimension 4)**.** *Let $G \subset SL_4(\mathbb{C})$ be a finite subgroup acting irreducibly. Then the singularity of $\mathbb{C}^4/G$ is not weakly-exceptional exactly when G is conjugate to one of the group actions described in Proposition 3.6, Lemma 3.8 and Lemma 3.12.*

Proof: Immediate by applying Theorem 2.49 and Lemma 3.7. □

3.3 Five-dimensional case

Let $G \subset \mathrm{SL}_5(\mathbb{C})$ be a finite irreducible subgroup. Recall that, by Theorem 2.51, the singularity induced by G is not weakly-exceptional if and only if G has a semi-invariant of degree at most 4. First of all, consider the types of group actions that exist in this dimension:

Proposition 3.14. *Let $q > 1$ be a prime number, and let $G \subset SL_q(\mathbb{C})$ be a finite irreducible subgroup. Then the action of G is either primitive or monomial.*

Proof: Given any system of imprimitivity for G, Proposition 2.5 implies that all the subspaces in that system must have the same dimension d, with $d|q$. Since q is prime, $d \in \{1, q\}$. If there exists a system with 1-dimensional subspaces, then the action of G is monomial. Otherwise, the action of G must be primitive. □

Since by Theorem 2.45, only the irreducible groups need to be considered, and since by Lemma 2.16, there are only finitely many primitive subgroups of $\mathrm{SL}_5(\mathbb{C})$, the main problem here comes from dealing with the monomial groups. The primitive groups will be considered in Lemma 3.23, which will be proven at the end of this section, since the proof can be streamlined by using a result about a monomial group. So for now, assume that the group G is monomial.

Lemma 3.15. *Assume $G \subset SL_q(\mathbb{C})$ is a finite irreducible monomial subgroup. Setting $G \cong D \rtimes T$ as in Proposition 2.7, there exists $\tau \in G \setminus D$ and a basis $e_1, \ldots, e_q$ for $\mathbb{C}^q$, such that $\tau^q = Id_G$, and τ acts by*

$$\tau(e_i) = e_{i+1} \ \forall i < q; \ \tau(e_q) = e_1$$

Proof: Since G is irreducible, T must be a transitive subgroup of $\mathbb{S}_q$ (by Proposition 2.8) and must thus contain a cycle of length q (since q is prime). Take $\tau \in \Gamma$, such that $\pi(\tau)$ is a generator of this cycle. Let $e_1 \in V_1$ be a non-zero vector. Then, renaming the V_i-s if necessary, $\tau^i(e_1) \in V_{i+1}$ (for $1 \leq i < q$). Set $e_i = \tau^{i-1}(e_1)$ $(2 \leq i \leq q)$. Clearly, $\tau(e_q) = \alpha e_1$ for some $\alpha \in \mathbb{C}$.

Since all the subspaces V_i are disjoint and one-dimensional, e_i must generate V_i, and so $e_1, \ldots, e_q$ must form a basis for $\mathbb{C}^q$. Also, since $g \in D = \ker \pi$, and τ permutes the subspaces V_i non-trivially, $\tau \notin D$. Since $\tau \in G \subseteq \mathrm{SL}_q(\mathbb{C})$ and q odd, one also observes that $\alpha = 1$, and so τ acts as stated above. □

Fix this element τ (and the corresponding basis for $\mathbb{C}^5$) throughout this section. This implies that the group T must be one of the subgroups of $\mathbb{S}_5$ containing

$\mathbb{Z}_5$. For the list of such subgroups, see Proposition 2.56. It is now necessary to look at the possibilities for D. First consider the case of D being trivial (either empty or central in $\mathrm{SL}_5(\mathbb{C})$):

Lemma 3.16. *If all the elements of D are scalar, then the singularity induced by G is not weakly-exceptional.*

Proof: In this case, G must be either one of the groups mentioned in Remark 2.56 or a central extension of one of them by $\mathbb{Z}_5$. On this list, the only groups that have irreducible 5-dimensional representations are $\mathbb{A}_5$, $\mathbb{S}_5$ and their central extensions by $\mathbb{Z}_5$. It is easy to see that all of these have semi-invariants of degree 2. □

From now on, one can assume that D contains a non-scalar element. Furthermore, in most cases it is possible to assume that D is generated by non-scalar elements of prime order:

Lemma 3.17. *Let $g \in D$ be a non-scalar element of order pq for some integers $p, q > 1$. Then either $p = 5$, or $\exists g' \in D$ a non-scalar element of order p.*

Proof: Set $g' = g^q$. Scalar elements in $\mathrm{SL}_5(\mathbb{C})$ have orders 1 or 5, so either $p = 5$ or g' is not a scalar. □

Since the only obstruction to the singularity induced by G being weakly exceptional are the G-semi-invariant polynomials of small degree, it is useful to limit the space of such possible polynomials. This can be done as follows:

Proposition 3.18. *Define the following monomials in 5 variables $x_1, \ldots, x_5$:*

$m_{1,1} = x_1$			
$m_{2,1} = x_1^2$	$m_{2,2} = x_1x_2$	$m_{2,3} = x_1x_3$	
$m_{3,1} = x_1^3$	$m_{3,2} = x_1^2x_2$	$m_{3,3} = x_1^2x_3$	$m_{3,4} = x_1^2x_4$
$m_{3,5} = x_1^2x_5$	$m_{3,6} = x_1x_2x_3$	$m_{3,7} = x_1x_2x_4$	
$m_{4,1} = x_1^4$	$m_{4,2} = x_1^3x_2$	$m_{4,3} = x_1^3x_3$	$m_{4,4} = x_1^3x_4$
$m_{4,5} = x_1^3x_5$	$m_{4,6} = x_1^2x_2^2$	$m_{4,7} = x_1^2x_3^2$	$m_{4,8} = x_1^2x_2x_3$
$m_{4,9} = x_1^2x_2x_4$	$m_{4,10} = x_1^2x_2x_5$	$m_{4,11} = x_1^2x_3x_4$	$m_{4,12} = x_1^2x_3x_5$
$m_{4,13} = x_1^2x_4x_5$	$m_{4,14} = x_1x_2x_3x_4$		

Then any polynomial f of degree at most 4 that is semi-invariant under the action of τ must be one of

$$\begin{array}{ll} A_1 \sum_{j=0}^{4} \omega^j \tau^j (m_{1,1}) & \sum_{i=1}^{3} \left[B_i \sum_{j=0}^{4} \omega^j \tau^j (m_{2,i})\right] \\ \sum_{i=1}^{7} \left[C_i \sum_{j=0}^{4} \omega^j \tau^j (m_{3,i})\right] & \sum_{i=1}^{14} \left[D_i \sum_{j=0}^{4} \omega^j \tau^j (m_{4,i})\right] \end{array}$$

where $A_1, B_i, C_i, D_i \in \mathbb{C}$ and ω is some (not necessarily primitive) fifth root of 1.

Proof: The polynomial f is semi-invariant under the action of τ, so set ω such that $\omega f = \tau(f)$. Have $\tau^5 = \mathrm{id}$, so $\omega^5 = 1$.

Any polynomial that is τ-semi-invariant and contains a monomial m must contain all the monomials from the τ-orbit of m. It is easy to check that the $m_{d,i}$

above are representatives of all orbits of degree $d \leq 4$ monomials in 5 variables, and the result follows. $\square$

From now on, assume that the singularity G induces is not weakly exceptional. This means that one of the polynomials above must be G-semi-invariant. This allows to limit the possibilities for the group G by considering the limitations some elements G might have put on the parameters A_i, B_i, C_i, D_i.

First look at how the elements of D act on these polynomials. Since D preserves the basis of $\mathbb{C}^5$, all the monomials are D-semi-invariant, so every $\tau-$invariant polynomial must be preserved. Applying $g = [p, a_1, \ldots, a_5]$ (p prime, $0 \leq a_i < p$, a_i not all equal), get:

Lemma 3.19. *For any $g = [n, a_1, \ldots, a_5] \in D$, a_i not all equal (i.e. g is not scalar), the following hold (replacing g by its scalar multiple if necessary) for some parameter $a \in \mathbb{Z}$ ($0 < a < n$):*

$$\begin{array}{l}
A_1 = 0 \\
\hline
B_1 = 0 \ \textit{or}\ n = 2 \\
B_2 = B_3 = 0 \\
\hline
C_1 = 0 \ \textit{or}\ n = 3 \\
C_2 = 0 \ \textit{or}\ g = [11, a, 4^3a, 4^6a, 4^9a, 4^{12}a] \\
C_3 = 0 \ \textit{or}\ g = [11, a, 4^4a, 4^8a, 4^{12}a, 4^{16}a] \\
C_4 = 0 \ \textit{or}\ g = [11, a, 4^1a, 4^2a, 4^3a, 4^4a] \\
C_5 = 0 \ \textit{or}\ g = [11, a, 4^2a, 4^4a, 4^6a, 4^8a] \\
C_6 = C_7 = 0 \\
\hline
D_1 = 0 \ \textit{or}\ n \in \{2, 4\} \\
D_2 = 0 \ \textit{or}\ g = [61, a, 34^2a, 34^4a, 34^6a, 34^8a] \\
D_3 = 0 \ \textit{or}\ g = [61, a, 34^1a, 34^2a, 34^3a, 34^4a] \\
D_4 = 0 \ \textit{or}\ g = [61, a, 34^4a, 34^8a, 34^{12}a, 34^{16}a] \\
D_5 = 0 \ \textit{or}\ g = [61, a, 34^3a, 34^6a, 34^9a, 34^{12}a] \\
D_6 = D_7 = 0 \ \textit{or}\ n = 2 \\
D_8 = 0 \ \textit{or}\ g = [11, a, 4^1a, 4^2a, 4^3a, 4^4a] \\
D_9 = 0 \ \textit{or}\ g = [11, a, 4^2a, 4^4a, 4^6a, 4^8a] \\
D_{10} = D_{11} = 0 \\
D_{12} = 0 \ \textit{or}\ g = [11, a, 4^3a, 4^6a, 4^9a, 4^{12}a] \\
D_{13} = 0 \ \textit{or}\ g = [11, a, 4^4a, 4^8a, 4^{12}a, 4^{16}a] \\
D_{14} = 0
\end{array}$$

Proof:

Note that since $g \in \mathrm{SL}_5(\mathbb{C})$, $\sum_i a_i = 0 \mod n$. Furthermore, by Lemma 3.17, for any prime p, either $p \not| n$, or there exists $g' \in D$ with (in the notation above) $n' = p$. With this in mind, have:

- If $A_1 \neq 0$, then
$$a_1 \equiv a_2 \equiv a_3 \equiv a_4 \equiv a_5 \mod n$$
Therefore, g a scalar.

- If $B_1 \neq 0$, then

$$2a_1 \equiv 2a_2 \equiv 2a_3 \equiv 2a_4 \equiv 2a_5 \mod n$$

Therefore, either $n = 2$ or g is scalar.

- If $B_2 \neq 0$, then

$$a_1 + a_2 \equiv a_2 + a_3 \equiv a_3 + a_4 \equiv a_4 + a_5 \equiv a_5 + a_1 \mod n$$

Thus,

$$a_1 \equiv a_2 \equiv a_3 \equiv a_4 \equiv a_5 \mod n$$

Therefore, g a scalar.

Applying the permutation (2 3 5 4) to the indices of $x_1, \ldots, x_5$ in this computation implies that either $B_3 = 0$ or g is scalar.

- If $C_1 \neq 0$, then

$$3a_1 \equiv 3a_2 \equiv 3a_3 \equiv 3a_4 \equiv 3a_5 \mod n$$

Therefore, either $n = 3$ or g is scalar.

- If $C_2 \neq 0$, then

$$2a_1 + a_2 \equiv 2a_2 + a_3 \equiv 2a_3 + a_4 \equiv 2a_4 + a_5 \equiv 2a_5 + a_1 \mod n$$

If $n = 2$, then $a_1 \equiv \ldots \equiv a_5 \mod n$, making g a scalar. So $2 \not| n$ and

$$\begin{aligned}
2a_1 &\equiv a_2 + a_3 \mod n \\
2a_2 &\equiv a_3 + a_4 \mod n \\
2a_3 &\equiv a_4 + a_5 \mod n \\
2a_4 &\equiv a_5 + a_1 \mod n \\
2a_5 &\equiv a_1 + a_2 \mod n
\end{aligned}$$

Since $a_1 + \cdots + a_5 \equiv 0 \mod n$, get

$$\begin{aligned}
3a_1 + 2a_3 &\equiv 0 \mod n \\
3a_2 + 2a_4 &\equiv 0 \mod n \\
3a_3 + 2a_5 &\equiv 0 \mod n \\
3a_4 + 2a_1 &\equiv 0 \mod n \\
3a_5 + 2a_2 &\equiv 0 \mod n
\end{aligned}$$

So:

$$\begin{aligned}
3a_1 &\equiv -2a_3 \mod n\\
3a_2 &\equiv -2a_4 \mod n\\
3a_3 &\equiv -2a_5 \mod n\\
3a_4 &\equiv -2a_1 \mod n\\
3a_5 &\equiv -2a_2 \mod n
\end{aligned}$$

Since 2 is not a factor of n, it is invertible modulo n. So:

$$\begin{aligned}
4a_1 &\equiv 2a_2 + 2a_3 \equiv 3a_3 + a_4 \mod n\\
8a_1 &\equiv 3(2a_3) + 2a_4 \equiv 5a_4 + 3a_5 \mod n\\
16a_1 &\equiv 5(2a_4) + 6a_5 \equiv 11a_5 + 5a_1 \mod n
\end{aligned}$$

One deduces that either $11|n$ or $a_1 \equiv a_5 \mod n$. By symmetry,

$$11a_1 \equiv 11a_2 \equiv 11a_3 \equiv 11a_4 \equiv 11a_5 \mod n$$

Therefore, either $n = 11$ or g is scalar. Since $3 \equiv 4 \cdot (-2) \mod 11$ and $4^5 \equiv 1 \mod 11$, it is possible to deduce that $g = [11, a, 4^3a, 4^6a, 4^9a, 4^{12}a]$.

Applying powers of the permutation (2 3 5 4) to the indices of $x_1, \dots, x_5$ in this computation gives:

- $C_3 \neq 0 \Longrightarrow g = [11, a, 4^4a, 4^8a, 4^{12}a, 4^{16}a]$
- $C_4 \neq 0 \Longrightarrow g = [11, a, 4^1a, 4^2a, 4^3a, 4^4a]$
- $C_5 \neq 0 \Longrightarrow g = [11, a, 4^2a, 4^4a, 4^6a, 4^8a]$

• If $C_6 \neq 0$, then

$$a_1+a_2+a_3 \equiv a_2+a_3+a_4 \equiv a_3+a_4+a_5 \equiv a_4+a_5+a_1 \equiv a_5+a_1+a_2 \mod n$$

Thus,
$$a_1 \equiv a_2 \equiv a_3 \equiv a_4 \equiv a_5 \mod n$$

Therefore, g a scalar.

Applying the permutation (2 3 5 4) to the indices of $x_1, \dots, x_5$ in this computation implies that either $C_7 = 0$ or g is scalar.

• If $D_1 \neq 0$, then

$$4a_1 \equiv 4a_2 \equiv 4a_3 \equiv 4a_4 \equiv 4a_5 \mod n$$

Therefore, either $n \in \{2, 4\}$, or g is scalar.

• If $D_2 \neq 0$, then

$$3a_1 + a_2 \equiv 3a_2 + a_3 \equiv 3a_3 + a_4 \equiv 3a_4 + a_5 \equiv 3a_5 + a_1 \mod n$$

If $n = 3$, then $a_1 \equiv \ldots \equiv a_5 \mod n$, making g a scalar. So $3 \not| n$ and

$$\begin{aligned} 3a_1 &\equiv 2a_2 + a_3 \mod n \\ 3a_2 &\equiv 2a_3 + a_4 \mod n \\ 3a_3 &\equiv 2a_4 + a_5 \mod n \\ 3a_4 &\equiv 2a_5 + a_1 \mod n \\ 3a_5 &\equiv 2a_1 + a_2 \mod n \end{aligned}$$

Since $a_1 + \cdots + a_5 \equiv 0 \mod n$, get

$$\begin{aligned} 0 &\equiv 2(a_1 + \cdots + a_5) \equiv \\ &\equiv 2a_1 + (2a_2 + a_3) + a_3 + (2a_4 + a_5) + a_5 \mod n \\ &\equiv 2a_1 + 3a_1 + a_3 + 3a_3 + a_5 \equiv 5a_1 + 4a_3 + a_5 \mod n \\ &\equiv 5a_1 + 4a_3 + (3a_3 - 2a_4) \equiv 5a_1 + 7a_3 - 2(3a_2 - 2a_3) \mod n \\ &\equiv 5a_1 + 11a_3 - 3(2a_2) \equiv 5a_1 + 11a_3 - 3(3a_1 - a_3) \equiv 14a_3 - 4a_1 \mod n \end{aligned}$$

giving $4a_1 \equiv 14a_3 \mod n$. Similarly, get:

$$\begin{aligned} 4a_1 &\equiv 14a_3 \mod n \\ 4a_2 &\equiv 14a_4 \mod n \\ 4a_3 &\equiv 14a_5 \mod n \\ 4a_4 &\equiv 14a_1 \mod n \\ 4a_5 &\equiv 14a_2 \mod n \end{aligned}$$

Since 3 is not a factor of n, it is invertible modulo n. So:

$$\begin{aligned} 9a_1 &\equiv 2(3a_2) + 3a_3 \equiv 7a_3 + 2a_4 \mod n \\ 27a_1 &\equiv 7(3a_3) + 6a_4 \equiv 20a_4 + 7a_5 \mod n \\ 81a_1 &\equiv 20(3a_4) + 21a_5 \equiv 61a_5 + 20a_1 \mod n \end{aligned}$$

One deduces that either $61|n$ or $a_1 \equiv a_5 \mod n$. By symmetry,

$$61a_1 \equiv 61a_2 \equiv 61a_3 \equiv 61a_4 \equiv 61a_5 \mod n$$

Therefore, either $n = 61$ or g is scalar. Since $14 \equiv 34 \cdot 4 \mod 61$ and $34^5 \equiv 1 \mod 61$, it is possible to deduce that $g = [61, a, 34^2 a, 34^4 a, 34^6 a, 34^8 a]$.

Applying powers of the permutation (2 3 5 4) to the indices of $x_1, \ldots, x_5$ in this computation gives:

- $D_3 \neq 0 \Longrightarrow g = [61, a, 34^1 a, 34^2 a, 34^3 a, 34^4 a]$
- $D_4 \neq 0 \Longrightarrow g = [61, a, 34^4 a, 34^8 a, 34^{12} a, 34^{16} a]$
- $D_5 \neq 0 \Longrightarrow g = [61, a, 34^3 a, 34^6 a, 34^9 a, 34^{12} a]$

- If $D_6 \neq 0$, then

$$2a_1 + 2a_2 \equiv 2a_2 + 2a_3 \equiv 2a_3 + 2a_4 \equiv 2a_4 + 2a_5 \equiv 2a_5 + 2a_1 \mod n$$

Thus,

$$2a_1 \equiv 2a_2 \equiv 2a_3 \equiv 2a_4 \equiv 2a_5 \mod n$$

Therefore, either $n = 2$ or g a scalar.

Applying the permutation (2 3 5 4) to the indices of $x_1, \ldots, x_5$ in this computation implies that if $D_7 \neq 0$, then either $n = 2$ or g is scalar.

- If $D_8 \neq 0$, then

$$\begin{aligned} 2a_1 + a_2 + a_3 &\equiv 2a_2 + a_3 + a_4 \equiv 2a_3 + a_4 + a_5 \\ &\equiv 2a_4 + a_5 + a_1 \equiv 2a_5 + a_1 + a_2 \mod n \end{aligned}$$

If $n = 2$, then

$$a_2 + a_3 \equiv a_3 + a_4 \equiv a_4 + a_5 \equiv a_5 + a_1 \equiv a_1 + a_2 \mod n$$

This implies $a_1 \equiv \ldots \equiv a_5 \mod n$, making g a scalar. So $2 \nmid n$ and

$$\begin{aligned} 2a_1 &\equiv a_2 + a_4 \mod n \\ 2a_2 &\equiv a_3 + a_5 \mod n \\ 2a_3 &\equiv a_4 + a_1 \mod n \\ 2a_4 &\equiv a_5 + a_2 \mod n \\ 2a_5 &\equiv a_1 + a_3 \mod n \end{aligned}$$

Since $a_1 + \cdots + a_5 \equiv 0 \mod n$, get

$$\begin{aligned} 0 &\equiv a_1 + \cdots + a_5 \equiv a_1 + (a_2 + a_4) + (a_3 + a_5) \mod n \\ &\equiv 3a_1 + 2a_2 \mod n \end{aligned}$$

giving $3a_1 \equiv -2a_2 \mod n$. Similarly, get:

$$\begin{aligned} 3a_1 &\equiv -2a_2 \mod n \\ 3a_2 &\equiv -2a_3 \mod n \\ 3a_3 &\equiv -2a_4 \mod n \\ 3a_4 &\equiv -2a_5 \mod n \\ 3a_5 &\equiv -2a_1 \mod n \end{aligned}$$

Since 2 is not a factor of n, it is invertible modulo n. So:

$$\begin{aligned} 4a_1 &\equiv 2a_2 + 2a_4 \equiv 2a_2 + (a_2 + a_5) \equiv 3a_2 + a_5 \mod n \\ 8a_1 &\equiv 3(a_3 + a_5) + 2a_5 \equiv 5a_5 + 3a_3 \mod n \\ 16a_1 &\equiv 5(a_1 + a_3) + 6a_3 \equiv 11a_3 + 5a_1 \mod n \end{aligned}$$

One deduces that either $11 | n$ or $a_1 \equiv a_3 \mod n$. By symmetry,

$$11a_1 \equiv 11a_2 \equiv 11a_3 \equiv 11a_4 \equiv 11a_5 \mod n$$

Therefore, either $n = 11$ or g is scalar. Since $3 \equiv 4 \cdot (-2) \mod 11$ and $4^5 \equiv 1 \mod 11$, it is possible to deduce that $g = [11, a, 4^1 a, 4^2 a, 4^3 a, 4^4 a]$.

Applying powers of the permutation (2 3 5 4) to the indices of $x_1, \ldots, x_5$ in this computation gives:

- $D_9 \neq 0 \Longrightarrow g = [11, a, 4^2 a, 4^4 a, 4^6 a, 4^8 a]$
- $D_{12} \neq 0 \Longrightarrow g = [11, a, 4^3 a, 4^6 a, 4^9 a, 4^{12} a]$
- $D_{13} \neq 0 \Longrightarrow g = [11, a, 4^4 a, 4^8 a, 4^{12} a, 4^{16} a]$

- If $D_{10} \neq 0$, then

$$\begin{aligned} 2a_1 + a_2 + a_5 &\equiv 2a_2 + a_3 + a_1 \equiv 2a_3 + a_4 + a_2 \\ &\equiv 2a_4 + a_5 + a_3 \equiv 2a_5 + a_1 + a_4 \mod n \end{aligned}$$

In particular,

$$\begin{aligned} 2a_1 + a_2 + a_5 &\equiv 2a_5 + a_1 + a_4 \mod n \\ 2a_3 + a_4 + a_2 &\equiv 2a_4 + a_5 + a_3 \mod n \end{aligned}$$

Therefore,

$$a_1 + a_2 \equiv a_4 + a_5 \equiv a_2 + a_3 \mod n$$

So $a_1 \equiv a_3 \mod n$. Proceeding symmetrically, get

$$a_1 \equiv a_2 \equiv a_3 \equiv a_4 \equiv a_5 \mod n$$

Thus g is scalar.

Applying the permutation (2 3 5 4) to the indices of $x_1, \ldots, x_5$ in this computation implies that either $D_{11} = 0$ or g is scalar.

- If $D_{14} \neq 0$, then

$$\begin{aligned} a_1 + a_2 + a_3 + a_4 &\equiv a_2 + a_3 + a_4 + a_5 \equiv a_3 + a_4 + a_5 + a_1 \\ &\equiv a_4 + a_5 + a_1 + a_2 \equiv a_5 + a_1 + a_2 + a_3 \mod n \end{aligned}$$

Thus,

$$a_1 \equiv a_2 \equiv a_3 \equiv a_4 \equiv a_5 \mod n$$

Therefore, g a scalar.

□

Corollary 3.20. *Let $G \subset SL_5(\mathbb{C})$ be a finite irreducible monomial group that induces a non-weakly-exceptional singularity. Then either $\|D\|$ or $\|D\|/5$ is in $\{2^k, 3^k, 11^k, 61^k\}$ for some positive integer k.*

Now it is necessary to look at the possible values the group T can take for the different isomorphism classes of D.

Proposition 3.21. *Let $G \subset SL_5(\mathbb{C})$ be a finite irreducible monomial group that induces a non-weakly-exceptional singularity, and $\exists g \in G$ an element of order 11 or 61. Then $G = D \rtimes \mathbb{Z}_5$ (with D as above).*

Proof: It is easy to see that $D \rtimes \mathbb{Z}_5 \subseteq G$. Assume the inequality is strict. Then by looking at the action of G on the polynomials, it is clear that $C_i, C_j \neq 0$ for some $2 \leq i \neq j \leq 5$. Then any elements of D must be of the form specified in Lemma 3.19. However, it is easy to see that an element being in two of the forms at the same time means (in the notation of Lemma 3.19) that $a = 0$, and so this is the identity element, leading to a contradiction. A similar argument works for the relevant D_i-s. □

Proposition 3.22. *Let G be a finite monomial group as described above preserving the polynomial*

$$\begin{aligned} h(x_1, \ldots, x_5) &= D_6 \left(x_1^2 x_2^2 + \omega x_2^2 x_3^2 + \omega^2 x_3^2 x_4^2 + \omega^3 x_4^2 x_5^2 + \omega^4 x_5^2 x_1^2\right) \\ &+ D_7 \left(x_1^2 x_3^2 + \omega x_2^2 x_4^2 + \omega^2 x_3^2 x_5^2 + \omega^3 x_4^2 x_1^2 + \omega^4 x_5^2 x_2^2\right) \end{aligned}$$

semi-invariant for some values of D_6, D_7 not both zero, and some ω a fifth root of 1. Then $\omega = 1$, and the polynomial $f(x_1, \ldots, x_5) = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2$ is also G-semi-invariant.

Proof: Decompose $G = D \rtimes T$, $\tau \in T$ as above. Lemma 3.19 implies that for any $g \in D$, g^2 is a scalar, and so such D also leaves f semi-invariant. Therefore, it remains to check that the representatives of generators of T leave f semi-invariant. This is obviously true if $T \cong \mathbb{Z}_5$ (then T is generated by the image of τ).

Therefore, it remains to show the proposition holds for $\mathbb{Z}_5 \subsetneq T \subseteq \mathbb{S}_5$. Looking at the subgroups of $\mathbb{S}_5$, this means $\mathbb{D}_{10} \subseteq T \subseteq \mathbb{S}_5$. In particular $\exists \delta \in G \setminus D$, such that the image of δ is (up to conjugation and choosing τ appropriately) $(2\ 5)(3\ 4) \in \mathbb{D}_{10} \subseteq T \subseteq \mathbb{S}_5$ Therefore, $\exists \lambda_i \in \mathbb{C} \setminus 0$ such that g is defined by $(x_1, x_2, x_3, x_4, x_5) \mapsto (\lambda_1 x_1, \lambda_5 x_5, \lambda_4 x_4, \lambda_3 x_3, \lambda_2 x_2)$.

Applying this to h and solving the resulting equations, get the equations $\lambda_2^2 = \lambda_1^2 \omega^4$, $\lambda_3^2 = \lambda_1^2 \omega^3$, $\lambda_4^2 = \lambda_1^2 \omega^3$, $\lambda_5^2 = \lambda_1^2 \omega$. By the definition of the semidirect product, have $\delta^2 \in D$, and so $\lambda_1^2 = C(-1)^{a_1}$, $\lambda_3 \lambda_4 = C(-1)^{a_3}$. This and the fact that (by construction) $\omega^5 = 1$ implies that $\omega = 1$, and hence implies that $\lambda_1^2 = \lambda_2^2 = \lambda_3^2 = \lambda_4^2 = \lambda_5^2$, making f semi-invariant under the action of δ.

Hence the proposition holds unless $\mathbb{D}_{10} \subsetneq T \subseteq \mathbb{S}_5$. Doing the same calculation (simplified, as $\omega = 1$) for the elements of $G \setminus D$ that are preimages of the elements $(1\ 2\ 3) \in \mathbb{A}_5 \subset \mathbb{S}_5$ and $(2\ 3\ 5\ 4) \in \mathbb{GA}(1, 5) \subset \mathbb{S}_5$ excludes the remaining 3 possibilities for T. □

To complete the classification in this dimension, one needs to consider the case of G being primitive. In that case, G must be isomorphic to one of the groups in Theorem 2.21. Using this classification, one can get the following result:

Lemma 3.23. *Let $G \subset SL_5(\mathbb{C})$ be a finite primitive subgroup. Then G gives rise to a weakly-exceptional singularity if and only if it contains a subgroup isomorphic to the Heisenberg group $\mathbb{H}$.*

Proof: Since there is a very small number of such groups (see Theorem 2.21), one can simply look at the low symmetric powers of their 5-dimensional irreducible representations. This gives:

- The actions of $\mathbb{A}_5$, $\mathbb{S}_5$, $\mathbb{A}_6$, $\mathbb{S}_6$ have semi-invariants of degree 2, since they are conjugate to sub- groups of $\mathrm{GL}_5(\mathbb{R})$
- The action of $\mathrm{PSL}_2(11)$ has a semi-invariant of degree 3, the Klein cubic threefold (see [1]).
- The action of $\mathrm{Sp}_4(\mathbb{F}_3)$ has a semi-invariant of degree 4, the Burkhardt quartic threefold (see [4]).
- If G contains the Heisenberg group $\mathbb{H}$, then G cannot have any semi-invariants of degree at most 4 (either apply Theorem 2.51 to [9, Theorem 1.15] or apply Lemma 3.19 to the (monomial) representations of H of dimension at most 5).

Comparing this with the list of finite primitive subgroups of $\mathrm{SL}_5(\mathbb{C})$ implies the result. □

Theorem 3.24 (Main theorem in dimension 5)**.** *Let $G \subset SL_5(\mathbb{C})$ be a finite subgroup acting irreducibly. Then the singularity of $\mathbb{C}^5/G$ is weakly-exceptional exactly when:*

1. *The action of G is primitive and G contains a subgroup isomorphic to the Heisenberg group of all unipotent 3×3 matrices over $\mathbb{F}_5$ (for a better classification of all such groups, see [26]).*
2. *The action of G is monomial (making $G \cong D \rtimes T$, with D an abelian group as above and T a transitive subgroup of $\mathbb{S}_5$), and none of the following hold:*
   - *D is central in $SL_5(\mathbb{C})$. In this case, G can be isomorphic to $\mathbb{A}_5$, $\mathbb{S}_5$, or their central extensions by $\mathbb{Z}_5$.*
   - *$\|G\| = 55$ or $55 \cdot 5$ with $\|D\| = 11$ or $11 \cdot 5$ resp., $T \cong \mathbb{Z}_5 \subset \mathbb{S}_5$, and there is a $k \in \mathbb{Z}$, $1 \leq k \leq 4$, such that D is generated by $\left[11, 1, 4^k, 4^{2k}, 4^{3k}, 4^{4k}\right]$ and (in the latter case) also the scalar element $\zeta_5 \cdot Id$. In this case, G is isomorphic to $\mathbb{Z}_{11} \rtimes \mathbb{Z}_5$ or $(\mathbb{Z}_5 \times \mathbb{Z}_{11}) \rtimes \mathbb{Z}_5$.*
   - *$\|G\| = 305$ or $305 \cdot 5$ with $\|D\| = 61$ or $61 \cdot 5$ resp., $T \cong \mathbb{Z}_5 \subset \mathbb{S}_5$, and there is a $k \in \mathbb{Z}$, $1 \leq k \leq 4$, such that D is generated by $\left[61, 1, 34^k, 34^{2k}, 34^{3k}, 34^{4k}\right]$ and (in the latter case) also the scalar element $\zeta_5 Id$. In this case, G is isomorphic to $\mathbb{Z}_{61} \rtimes \mathbb{Z}_5$ or $(\mathbb{Z}_5 \times \mathbb{Z}_{61}) \rtimes \mathbb{Z}_5$.*
   - *There exists some $d \in \{2, 3, 4\}$ and ω with $\omega^5 = 1$, such that:*
     - *$\forall g \in D$, g^d is a scalar.*
     - *$\|D\| \in \left\{d^k, 5 \cdot d^k\right\}$ (depending on whether D contains any non-trivial scalar elements) with $1 \leq k \leq 4$.*
     - *The polynomial $x_1^d + \omega x_2^d + \omega^2 x_3^d + \omega^3 x_4^d + \omega^4 x_5^d$ is G-semi-invariant.*

Proof: If the action of G is primitive, then the result follows from Lemma 3.23; consider the case of G being irreducible monomial. Decompose $G = D \rtimes T$ as above.

If D is central in $\mathrm{SL}_5(\mathbb{C})$, then the result comes from Lemma 3.16. Otherwise see Lemma 3.19 for the list of possible conjugacy classes of D. For the relevant isomorphism classes of T, see Lemma 3.21 and Proposition 3.22. $\square$

Chapter 4

Weakly exceptional singularities in prime dimension

This chapter is dedicated to examining weakly exceptional singularity in prime-dimensional spaces in general. Chapter 3 contained examples of such singularities, namely classifications of those from dimensions 3 and 5. These suggest what one can expect from a classification in a general prime dimension.

The first observation one should make is that although one might expect a complete list in any given prime dimension of the groups inducing singularities that are not weakly exceptional, one should not expect a general list covering all prime dimensions. The reasons for this can be observed in dimension 5 (see Theorem 3.24), where in the monomial case the cycles of length 11 and 61 arise. Although it is clear that groups with longer cycles will arise (through the interplay of powers in low-degree monomials), there does not seem to be a rule, which would predict these in advance. Moreover, getting the list of relevant primitive groups would require classifying primitive groups in arbitrarily high dimension, which is a long-standing program in representation theory that is not likely to ever be completed.

The second observation is that in the prime dimensions covered in Chapter 3 (unlike dimension 4), the list of such groups is finite. This (as well as the similarity of the weak exceptionality criteria in these dimensions) suggests that in a general prime dimension, this list is finite too. The first part of this chapter is dedicated to proving this conjecture. The second part suggests an algorithm for producing this list in any given prime dimension. as an example of an immediate application, it also gives a better bound on the types of groups the list would contain in dimension 7.

4.1 Finiteness of counterexamples

Since one is only interested in the finiteness of the list, one can simplify the problem by disregarding the primitive group actions: by Jordan's Theorem (see Lemma 2.16), there are only finitely many primitive subgroups of $\mathrm{SL}_N(\mathbb{C})$ in any dimension N, so they would add at most finitely many elements to the list.

For this chapter, let $q \geq 3$ be a prime number, and $\Gamma \subset \mathrm{SL}_q(\mathbb{C})$ be a finite irreducible imprimitive subgroup, such that the singularity of $\mathbb{C}^q/\Gamma$ is not weakly exceptional.

Corollary 4.1. *There exists a subgroup $G = D \rtimes \mathbb{Z}_q \subseteq \Gamma$ generated by D and τ. The singularity of $\mathbb{C}^q/G$ is not weakly exceptional, and $\|\Gamma\| \leq (q-1)!\|G\|$.*

Proof: Take G generated by D and the element $\tau \in \Gamma$ obtained in Lemma 3.15. Clearly, $G \subseteq \Gamma$ and, looking at the action of τ, $G \cong D \rtimes \mathbb{Z}_q$. Let $\bar{\Gamma}$ and $\bar{G}$ be projections of Γ and G (respectively) to $\mathrm{PGL}_q(\mathbb{C})$. Then $\bar{G} \subseteq \bar{\Gamma}$, and any $\bar{\Gamma}$-invariant variety is also $\bar{G}$-invariant. Thus, using Theorem 2.52, the singularity induced by G is not weakly exceptional. Finally,

$$\|\Gamma\| \leq \frac{\|\mathbb{S}_q\|}{\|\mathbb{Z}_q\|}\|G\| = (q-1)!\|G\|$$

□

From now on, fix the group G constructed above, the subgroup $D \subset G$, the element $\tau \in G$ and the basis $e_1, \ldots, e_q$ for $\mathbb{C}^q$ constructed in Lemma 3.15. It is now necessary to obtain a specialised criterion for determining whether or not such groups induce a weakly exceptional singularity.

Proposition 4.2. *Any irreducible representation of G (given above) over $\mathbb{C}$ is either 1-dimensional or q-dimensional.*

Proof: Directly by Lemma 2.10: here, $A = D$, $(G : D) = q$, which is only divisible by 1 or itself. □

Lemma 4.3 (generalising [9, Theorem 3.4])**.** *Let q be an odd prime and assume $G \subset SL_q(\mathbb{C})$ is a finite imprimitive subgroup isomorphic to $A \rtimes \mathbb{Z}_q$ for some abelian A. Then the singularity of $\mathbb{C}^q/G$ is not weakly exceptional if and only if G has a (non-constant) semi-invariant of degree $d < q$.*

Proof: If G does have a semi-invariant of degree at most $q-1$, then the singularity is not weakly exceptional by Theorem 2.47. Suppose that G does not have any such semi-invariants, but the singularity is not weakly exceptional.

Then, by Theorem 2.52, there exists a $\bar{G}$-invariant irreducible normal Fano type variety $V \subset \mathbb{P}^{q-1}$, such that $\deg V \leq \begin{pmatrix} q-1 \\ \dim V \end{pmatrix}$ and $h^i(V, \mathcal{O}_V(m)) = 0$ $\forall i \geq 1$ $\forall m \geq 0$ (where $\mathcal{O}_V(m) = \mathcal{O}_V \otimes \mathcal{O}_{\mathbb{P}^{q-1}}(m)$).

Let $n = \dim V$. Then, since G has no semi-invariants of degree less than q, have $n \leq q-2$. Let $\mathcal{I}_V$ be the ideal sheaf of V. Then

$$h^0(V, \mathcal{O}_V(m)) = h^0\left(\mathbb{P}^{q-1}, \mathcal{O}_{\mathbb{P}^{q-1}}(m)\right) - h^0\left(\mathbb{P}^{q-1}, \mathcal{I}_V(m)\right)$$

For instance, $h^0(V, \mathcal{O}_V) = 1$.

Take any $m \in \mathbb{Z}$ with $0 < m < q$. Let $W_m = H^0(\mathbb{P}^{q-1}, \mathcal{I}_V(m))$. This is a linear representation of G, so $q | \dim W_m$ (by Proposition 4.2, as G has no semi-invariants of degree $m < q$). Since $q | h^0\left(\mathbb{P}^{q-1}, \mathcal{O}_{\mathbb{P}^{q-1}}(m)\right)$,

$$h^0(V, \mathcal{O}_V(m)) \equiv 0 \mod q$$

Since $h^0(V, \mathcal{O}_V(t)) = \chi(V, \mathcal{O}_V(t))$ for any integer $t \geq 0$, there exist integers $a_0, \ldots, a_n$, such that

$$h^0(V, \mathcal{O}_V(t)) = P(t) = a_n t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + a_0$$

Consider $P(t)$ as a polynomial over $\mathbb{Z}_q$. Since

$$P(m) = h^0(V, \mathcal{O}_V(m)) \equiv 0 \mod q$$

whenever $0 < m < q$, $P(t)$ has at least $q-1$ roots over $\mathbb{Z}_q$. But $\deg P \leq n \leq q-2$, so $P(t)$ must be the zero polynomial over $\mathbb{Z}_q$. In particular, $a_0 \equiv 0 \mod q$. On the other hand, $a_0 = P(0) = h^0(V, \mathcal{O}_V) = 1 \not\equiv 0 \mod q$, leading to a contradiction. $\square$

Now let $f(x_1, \ldots, x_q)$ be a semi-invariant of G of degree $d < q$ from Lemma 4.3. Using the chosen basis, let

$$m(x_1, \ldots, x_q) = x_1^{a_1} x_2^{a_2} \cdots x_q^{a_q}$$

be a monomial contained in f (for some $a_i \in \mathbb{Z}_{\geq 0}$). Then $\sum_i a_i = d$ and $\sum_{i=0}^{q} \lambda^i \tau^i(m)$ is a semi-invariant of G whenever $\lambda^q = 1$. So, without loss of generality, assume

$$f(x_1, \ldots, x_q) = \left[m + \lambda\tau(m) + \cdots + \lambda^{q-1}\tau^{q-1}(m)\right](x_1, \ldots, x_q)$$

Note that all the a_i are non-negative integers, not all zero, and $0 < \sum_i a_i = d < q$.

This semi-invariant can now be exploited to obtain a bound for the possible size of D. To do this, the following lemma is necessary:

Lemma 4.4. *Consider the following q by q matrix with integer coefficients:*

$$M = \begin{pmatrix} a_1 & a_2 & \cdots & a_{q-1} & a_q \\ a_q & a_1 & \cdots & a_{q-2} & a_{q-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_3 & a_4 & \cdots & a_1 & a_2 \\ a_2 & a_3 & \cdots & a_q & a_1 \end{pmatrix}$$

The determinant of M is not zero.

Proof: Consider the matrix M over $\mathbb{C}$, and assume $\det M = 0$. Then one of the eigenvalues of M must be zero. So, by Lemma 2.55,

$$a_1 + \omega a_2 + \omega^2 a_3 + \ldots + \omega^{q-1} a_q = 0$$

for some ω with $\omega^q = 1$. Since all the a_i-s are non-negative integers, this is a sum of exactly $d = \sum_{i=1}^{q} a_i$ q-th roots of unity. So, by Theorem 2.53, d must be a sum of the prime factors of q. But, by the initial assumptions, q is prime, and $0 < d < q$, producing a contradiction. $\square$

This allows to bound the size of cyclic subgroups of D:

Lemma 4.5. *Let $g \in D$, and let n be the smallest positive integer, such that g^n is a scalar matrix. Then $n < q^{2q+1}$.*

Proof: Assume $n > 1$. Since $g \in G \subset \mathrm{SL}_q(\mathbb{C})$, $g^n = \zeta_q \mathrm{I}_q$, where ζ_q is a q-th root of 1 and I_q is the identity matrix. Then, since all the elements of D are diagonal matrices,

$$g = \zeta_q^{\beta_0} \begin{pmatrix} \zeta_n^{\beta_1} & & \\ & \ddots & \\ & & \zeta_n^{\beta_q} \end{pmatrix}$$

where $\beta_i \in \mathbb{Z}$, not all zero, with $0 \leq \beta_i < n \ \forall i > 0$; $0 \leq \beta_0 < q$. Since n was taken to be minimal, the highest common factor of $\{n, \beta_1, \dots, \beta_q\}$ is 1.

Now consider the polynomial f of degree $d < q$ described above. Since we know $g \in G$, $g(f) = \lambda f$ for some $\lambda \in \mathbb{C}$. Since $g^{nq} = \mathrm{I}_q$ and all the monomials are g-semi-invariant, $\lambda = \zeta_q^{\beta_0} \zeta_n^C$, some $C \in \mathbb{Z}$. This is equivalent to:

$$\begin{aligned} C &\equiv a_1\beta_1 + a_2\beta_2 + \cdots + a_{q-1}\beta_{q-1} + a_q\beta_q \mod n \\ &\equiv a_1\beta_2 + a_2\beta_3 + \cdots + a_{q-1}\beta_q + a_q\beta_1 \mod n \\ &\equiv a_1\beta_3 + a_2\beta_4 + \cdots + a_{q-1}\beta_1 + a_q\beta_2 \mod n \\ &\quad \dots \\ &\equiv a_1\beta_q + a_2\beta_1 + \cdots + a_{q-1}\beta_{q-2} + a_q\beta_{q-1} \mod n \end{aligned}$$

This can be rewritten as

$$M(\beta_1, \dots, \beta_q)^T \equiv C(1, \dots, 1)^T \mod n$$

where M is the matrix from Lemma 4.4). However, since $\sum_{i=1}^q a_i = d$, M also satisfies

$$M(1, \dots, 1)^T = d(1, \dots, 1)^T$$

Take $v = d(\beta_1, \dots, \beta_q)^T - C(1, \dots, 1)^T$. By linearity, $Mv \equiv 0 \mod n$. Multiplying both sides by the adjugate matrix of M, get:

$$\begin{aligned} (d\beta_1 - C)\det M &\equiv 0 \mod n \\ (d\beta_2 - C)\det M &\equiv 0 \mod n \\ &\dots \\ (d\beta_q - C)\det M &\equiv 0 \mod n \end{aligned}$$

Therefore,

$$d\beta_1 \det M \equiv d\beta_2 \det M \equiv \cdots \equiv d\beta_q \det M \equiv C \det M \mod n$$

This implies that $g^{d \det M}$ is a scalar matrix. By assumption, $0 < d < q$ (in $\mathbb{Z}$), and, by Lemma 4.4, $\det M \neq 0$ (in $\mathbb{Z}$), so $\|d \det M\| = Kn$ for some positive integer K. Thus, $n \leq \|d \det M\| \leq q\|detM\|$.

Now look at the entries $M_{i,j}$ of the matrix M. Since $0 \leq a_k \leq d < q$ for all k,

$\|M_{i,j}\| \le d < q$. Thus,

$$n \le q\|\det M\| \le q\left(q \max_{i,j} \|M_{i,j}\|\right)^q < q^{2q+1}$$

$\square$

Corollary 4.6. *Let $\mathbb{Z}_m \subseteq D$. Then $m \le q^{2q+2}$.*

Proof: Take g a generator of $\mathbb{Z}_m \subseteq D$. Then for some $n \le q^{2q+1}$, g^n is a scalar matrix in $\mathrm{SL}_q(\mathbb{C})$. Therefore, $g^{qn} = \mathrm{id}$. So

$$m \le qn \le q \cdot q^{2q+1} = q^{2q+2}$$

$\square$

Lemma 4.7. *Let $(\mathbb{Z}_m)^k \subseteq D \subset G \subseteq \Gamma \subset SL_q(\mathbb{C})$. Then $k \le q$.*

Proof: Let $g_1, \ldots, g_k$ be a minimal set of generators of $(\mathbb{Z}_m)^k \subseteq D$. Then for every $i > 1$, $g_i \notin <g_1, \ldots, g_{i-1}>$. Let ζ_m be a primitive m-th root of 1. Then all the g_i are diagonal matrices with some powers of ζ_m as diagonal entries. But any matrix in $\mathrm{SL}_q(\mathbb{C})$ has exactly q diagonal entries, so at most q such g_i-s can be chosen. Therefore, $k \le q$. $\square$

Corollary 4.8. $D \subseteq \bigotimes_{i=0}^{q^{2q+2}} (\mathbb{Z}_i)^q$.

Proof: Immediate from Corollary 4.6 and Lemma 4.7. $\square$

Theorem 4.9. *Given $q > 3$, there are at most finitely many finite irreducible monomial groups $\Gamma \subseteq SL_q(\mathbb{C})$, such that the singularity of $\mathbb{C}^q/\Gamma$ is not weakly exceptional.*

Proof: Let Γ be such a group. Then by Corollary 4.1, there exists $G \subseteq \Gamma$, such that $G \cong D \rtimes \mathbb{Z}_q$ and $\|\Gamma\| \le (q-1)!\|G\|$. By Corollary 4.8, $D \subseteq \bigotimes_{i=0}^{q^{2q+2}} (\mathbb{Z}_i)^q$, so there are at most finitely many such group D. It follows that there are at most finitely many such groups G, and hence at most finitely many such groups Γ. $\square$

Now, for any prime $q \ge 2$, consider the set of all finite irreducible subgroups $\Gamma \subset \mathrm{SL}_q(\mathbb{C})$ such that the singularity induced by Γ is not weakly exceptional. If $q = 2$, no such groups exist (see Example 2.22), so can assume $q \ge 3$. By Jordan's lemma (Lemma 2.16), there are only finitely many primitive subgroups of $\mathrm{SL}_q(\mathbb{C})$. Thus, one can assume Γ is imprimitive, and therefore, by Proposition 3.14, monomial. But Theorem 4.9, says that there are only finitely many such groups Γ. This proves the main result of this thesis:

Theorem 4.10 (Main theorem)**.** *Let q be a positive prime integer. Then there are at most finitely many finite irreducible subgroups $\Gamma \subset SL_q(\mathbb{C})$, such that the singularity induced by Γ is not weakly exceptional.*

4.2 Computational algorithm

The proof above gives rise to an algorithm, that, for any prime dimension q, can list all the imprimitive irreducible subgroups of $\mathrm{SL}_q(\mathbb{C})$ that give rise to singularities that are not weakly exceptional. To begin, decompose the group $G = D \rtimes T$, with D consisting of diagonal matrices and $T \subseteq \mathbb{S}_q$. As before, first concentrate on the case of $T \cong \mathbb{Z}_q$ to find all the possible isomorphism classes for D. Once this is done, one can look into the other values for T.

This can be written up as the following algorithm:

1. Run through all possible degrees $d \in \{2, \ldots, q-1\}$ of an invariant.
2. Generate the possible subdivisions of d as a sum of integers $b_1, \ldots, b_q$, such that $b_1 \geq b_2 \geq \ldots \geq b_q \geq 0$.
3. For each of the subdivisions:
   (a) Generate the possible monomials: Take q-tuples $(a_1, \ldots, a_q)$, such that for some $\sigma \in \mathbb{S}_q$, $a_i = b_{\sigma i}$. This corresponds to the monomial $x_1^{a_1} x_2^{a_2} \ldots x_q^{a_q}$. To optimize the computation, exclude those elements σ that can be obtained by multiplying a previously considered element by an element of a q-cycle in $\mathbb{S}_q$: the equations below consider the $\mathbb{Z}_q$-orbits of the monomials, so only one representative of each orbit needs to be considered.
   (b) For each such q-tuple, construct the matrix M from Lemma 4.4. Let m_σ be its determinant.
   (c) Then the group D may have a cycle of length $n_\sigma = q m_\sigma d$.
   (d) Furthermore, all the elements of D are (in the chosen basis) diagonal matrices with diagonal entries $\zeta_{n_\sigma}^{x_i}$, where x_i are the components of an integer vector x satisfying the equations:
   $$\begin{aligned} dMx &\equiv \alpha u^T \mod n_\sigma \\ u \cdot x &\equiv 0 \mod n_\sigma \end{aligned}$$
   where $u = (1, \ldots, 1)$ is a vector of length q, and α is some integer.
   (e) For every such D (acceptable for a permutation σ above), construct the possible subgroups $T \subseteq \mathbb{S}_q$: an element $\tau' \in \mathbb{S}_q$ can be contained in T exactly when D is also acceptable for the permutation $\sigma\tau' \in \mathbb{S}_q$ (for the same subdivision).

At first glance, this algorithm seems unrealistic to implement due to the high theoretically possible number of elements of the group D (as seen from Corollary 4.8). However, in practice (for any given q) even the initial steps of the algorithm tend to lower this bound very significantly. This bound is regulated by the possible values for the integers n_σ, and, as an illustration, they all have been computed in the case of $q = 7$. The results are displayed in Tables 4.1—4.6.

Looking at these tables, one can deduce that $D \subseteq \mathbb{Z}_7 \times (\mathbb{Z}_{n \cdot d})^6$, where the values of n and d are as follows:

Table 4.1: Results for degree 2 monomials

Monomial	Determinant	Factors	Monomial	Determinant	Factors
Subdivision "$2+0+0+0+0+0+0$"					
x_1^2	-64	2^6			
Subdivision "$1+1+0+0+0+0+0$"					
x_1x_2	-1		x_1x_3	-1	
x_1x_4	-1				

Table 4.2: Results for degree 3 monomials

Monomial	Determinant	Factors	Monomial	Determinant	Factors
Subdivision "$3+0+0+0+0+0+0$"					
x_1^3	-729	3^6			
Subdivision "$2+1+0+0+0+0+0$"					
$x_1^2x_2$	-43	43	$x_1^2x_3$	-43	43
$x_1^2x_4$	-43	43	$x_1^2x_5$	-43	43
$x_1^2x_6$	-43	43	$x_1^2x_7$	-43	43
Subdivision "$1+1+1+0+0+0+0$"					
$x_1x_2x_3$	-1		$x_1x_2x_4$	-8	2^3
$x_1x_2x_5$	-1		$x_1x_2x_6$	-8	2^3
$x_1x_3x_5$	-1				

Table 4.3: Results for degree 4 monomials

Monomial	Determinant	Factors	Monomial	Determinant	Factors
	Subdivision "$4+0+0+0+0+0+0$"				
x_1^4	-4096	2^{12}			
	Subdivision "$3+1+0+0+0+0+0$"				
$x_1^3x_2$	-547	547	$x_1^3x_3$	-547	547
$x_1^3x_4$	-547	547	$x_1^3x_5$	-547	547
$x_1^3x_6$	-547	547	$x_1^3x_7$	-547	547
	Subdivision "$2+2+0+0+0+0+0$"				
$x_1^2x_2^2$	-64	2^6	$x_1^2x_3^2$	-64	2^6
$x_1^2x_4^2$	-64	2^6			
	Subdivision "$2+1+1+0+0+0+0$"				
$x_1^2x_2x_3$	-29	29	$x_1^2x_2x_4$	-71	71
$x_1^2x_2x_5$	-29	29	$x_1^2x_2x_6$	-71	71
$x_1^2x_2x_7$	-1		$x_1^2x_3x_4$	-71	71
$x_1^2x_3x_5$	-29	29	$x_1^2x_3x_6$	-1	
$x_1^2x_3x_7$	-71	71	$x_1^2x_4x_5$	-1	
$x_1^2x_4x_6$	-29	29	$x_1^2x_4x_7$	-29	29
$x_1^2x_5x_6$	-71	71	$x_1^2x_5x_7$	-71	71
$x_1^2x_6x_7$	-29	29			
	Subdivision "$1+1+1+1+0+0+0$"				
$x_1x_2x_3x_4$	-1		$x_1x_2x_3x_5$	-8	2^3
$x_1x_2x_3x_6$	-8	2^3	$x_1x_2x_4x_5$	-1	
$x_1x_2x_4x_6$	-1				

Table 4.4: Results for degree 5 monomials

Monomial	Determinant	Factors	Monomial	Determinant	Factors
		Subdivision "$5+0+0+0+0+0+0$"			
x_1^5	-15625	5^6			
		Subdivision "$4+1+0+0+0+0+0$"			
$x_1^4x_2$	-3277	$29 \cdot 113$	$x_1^4x_3$	-3277	$29 \cdot 113$
$x_1^4x_4$	-3277	$29 \cdot 113$	$x_1^4x_5$	-3277	$29 \cdot 113$
$x_1^4x_6$	-3277	$29 \cdot 113$	$x_1^4x_7$	-3277	$29 \cdot 113$
		Subdivision "$3+2+0+0+0+0+0$"			
$x_1^3x_2^2$	-463	463	$x_1^3x_3^2$	-463	463
$x_1^3x_4^2$	-463	463	$x_1^3x_5^2$	-463	463
$x_1^3x_6^2$	-463	463	$x_1^3x_7^2$	-463	463
		Subdivision "$3+1+1+0+0+0+0$"			
$x_1^3x_2x_3$	-421	421	$x_1^3x_2x_4$	-568	$2^3 \cdot 71$
$x_1^3x_2x_5$	-421	421	$x_1^3x_2x_6$	-568	$2^3 \cdot 71$
$x_1^3x_2x_7$	-169	13^2	$x_1^3x_3x_4$	-568	$2^3 \cdot 71$
$x_1^3x_3x_5$	-421	421	$x_1^3x_3x_6$	-169	13^2
$x_1^3x_3x_7$	-568	$2^3 \cdot 71$	$x_1^3x_4x_5$	-169	13^2
$x_1^3x_4x_6$	-421	421	$x_1^3x_4x_7$	-421	421
$x_1^3x_5x_6$	-568	$2^3 \cdot 71$	$x_1^3x_5x_7$	-568	$2^3 \cdot 71$
$x_1^3x_6x_7$	-421	421			
		Subdivision "$2+2+1+0+0+0+0$"			
$x_1^2x_2^2x_3$	-29	29	$x_1^2x_2^2x_4$	-197	197
$x_1^2x_2^2x_5$	-1		$x_1^2x_2^2x_6$	-197	197
$x_1^2x_2^2x_7$	-29	29	$x_1^2x_2x_3^2$	-1	
$x_1^2x_2x_4^2$	-197	197	$x_1^2x_2x_5^2$	-29	29
$x_1^2x_2x_6^2$	-197	197	$x_1^2x_3^2x_5$	-29	29
$x_1^2x_3^2x_6$	-29	29	$x_1^2x_3^2x_7$	-197	197
$x_1^2x_3x_4^2$	-197	197	$x_1^2x_3x_5^2$	-1	
$x_1^2x_4^2x_7$	-29	29			
		Subdivision "$2+1+1+1+0+0+0$"			
$x_1^2x_2x_3x_4$	-43	43	$x_1^2x_2x_3x_5$	-64	2^6
$x_1^2x_2x_3x_6$	-29	29	$x_1^2x_2x_3x_7$	-8	2^3
$x_1^2x_2x_4x_5$	-8	2^3	$x_1^2x_2x_4x_6$	-43	43
$x_1^2x_2x_4x_7$	-29	29	$x_1^2x_2x_5x_6$	-43	43
$x_1^2x_2x_5x_7$	-29	29	$x_1^2x_2x_6x_7$	-8	2^3
$x_1^2x_3x_4x_5$	-29	29	$x_1^2x_3x_4x_6$	-8	2^3
$x_1^2x_3x_4x_7$	-43	43	$x_1^2x_3x_5x_6$	-8	2^3
$x_1^2x_3x_5x_7$	-43	43	$x_1^2x_3x_6x_7$	-29	29
$x_1^2x_4x_5x_6$	-29	29	$x_1^2x_4x_5x_7$	-8	2^3
$x_1^2x_4x_6x_7$	-64	2^6	$x_1^2x_5x_6x_7$	-43	43
		Subdivision "$1+1+1+1+1+0+0$"			
$x_1x_2x_3x_4x_5$	-1		$x_1x_2x_3x_4x_6$	-1	
$x_1x_2x_3x_5x_6$	-1				

Table 4.5: Results for degree 6 monomials, part 1

Monomial	Determinant	Factors	Monomial	Determinant	Factors
		Subdivision "6 + 0 + 0 + 0 + 0 + 0 + 0"			
x_1^6	-46656	$2^6 \cdot 3^6$			
		Subdivision "5 + 1 + 0 + 0 + 0 + 0 + 0"			
$x_1^5x_2$	-13021	$29 \cdot 449$	$x_1^5x_3$	-13021	$29 \cdot 449$
$x_1^5x_4$	-13021	$29 \cdot 449$	$x_1^5x_5$	-13021	$29 \cdot 449$
$x_1^5x_6$	-13021	$29 \cdot 449$	$x_1^5x_7$	-13021	$29 \cdot 449$
		Subdivision "4 + 2 + 0 + 0 + 0 + 0 + 0"			
$x_1^4x_2^2$	-2752	$2^6 \cdot 43$	$x_1^4x_3^2$	-2752	$2^6 \cdot 43$
$x_1^4x_4^2$	-2752	$2^6 \cdot 43$	$x_1^4x_5^2$	-2752	$2^6 \cdot 43$
$x_1^4x_6^2$	-2752	$2^6 \cdot 43$	$x_1^4x_7^2$	-2752	$2^6 \cdot 43$
		Subdivision "4 + 1 + 1 + 0 + 0 + 0 + 0"			
$x_1^4x_2x_3$	-2689	2689	$x_1^4x_2x_4$	-3053	$43 \cdot 71$
$x_1^4x_2x_5$	-2689	2689	$x_1^4x_2x_6$	-3053	$43 \cdot 71$
$x_1^4x_2x_7$	-1681	41^2	$x_1^4x_3x_4$	-3053	$43 \cdot 71$
$x_1^4x_3x_5$	-2689	2689	$x_1^4x_3x_6$	-1681	41^2
$x_1^4x_3x_7$	-3053	$43 \cdot 71$	$x_1^4x_4x_5$	-1681	41^2
$x_1^4x_4x_6$	-2689	2689	$x_1^4x_4x_7$	-2689	2689
$x_1^4x_5x_6$	-3053	$43 \cdot 71$	$x_1^4x_5x_7$	-3053	$43 \cdot 71$
$x_1^4x_6x_7$	-2689	2689			
		Subdivision "3 + 3 + 0 + 0 + 0 + 0 + 0"			
$x_1^3x_2^3$	-729	3^6	$x_1^3x_3^3$	-729	3^6
$x_1^3x_4^3$	-729	3^6			
		Subdivision "3 + 2 + 1 + 0 + 0 + 0 + 0"			
$x_1^3x_2^2x_3$	-379	379	$x_1^3x_2^2x_4$	-757	757
$x_1^3x_2^2x_5$	-211	211	$x_1^3x_2^2x_6$	-841	29^2
$x_1^3x_2^2x_7$	-43	43	$x_1^3x_2x_3^2$	-211	211
$x_1^3x_2x_4^2$	-841	29^2	$x_1^3x_2x_5^2$	-379	379
$x_1^3x_2x_6^2$	-757	757	$x_1^3x_2x_7^2$	-43	43
$x_1^3x_3^2x_4$	-841	29^2	$x_1^3x_3^2x_5$	-379	379
$x_1^3x_3^2x_6$	-43	43	$x_1^3x_3^2x_7$	-757	757
$x_1^3x_3x_4^2$	-757	757	$x_1^3x_3x_5^2$	-211	211
$x_1^3x_3x_6^2$	-43	43	$x_1^3x_3x_7^2$	-841	29^2
$x_1^3x_4^2x_5$	-43	43	$x_1^3x_4^2x_6$	-211	211
$x_1^3x_4^2x_7$	-379	379	$x_1^3x_4x_5^2$	-43	43
$x_1^3x_4x_6^2$	-379	379	$x_1^3x_4x_7^2$	-211	211
$x_1^3x_5^2x_6$	-757	757	$x_1^3x_5^2x_7$	-841	29^2
$x_1^3x_5x_6^2$	-841	29^2	$x_1^3x_5x_7^2$	-757	757
$x_1^3x_6^2x_7$	-211	211	$x_1^3x_6x_7^2$	-379	379

Table 4.6: Results for degree 6 monomials, part 2

Monomial	Determinant	Factors	Monomial	Determinant	Factors
	Subdivision	"$3+1+1+1+0+0+0$"			
$x_1^3x_2x_3x_4$	-463	463	$x_1^3x_2x_3x_5$	-512	2^9
$x_1^3x_2x_3x_6$	-232	$2^3\cdot 29$	$x_1^3x_2x_3x_7$	-197	197
$x_1^3x_2x_4x_5$	-197	197	$x_1^3x_2x_4x_6$	-463	463
$x_1^3x_2x_4x_7$	-232	$2^3\cdot 29$	$x_1^3x_2x_5x_6$	-463	463
$x_1^3x_2x_5x_7$	-232	$2^3\cdot 29$	$x_1^3x_2x_6x_7$	-197	197
$x_1^3x_3x_4x_5$	-232	$2^3\cdot 29$	$x_1^3x_3x_4x_6$	-197	197
$x_1^3x_3x_4x_7$	-463	463	$x_1^3x_3x_5x_6$	-197	197
$x_1^3x_3x_5x_7$	-463	463	$x_1^3x_3x_6x_7$	-232	$2^3\cdot 29$
$x_1^3x_4x_5x_6$	-232	$2^3\cdot 29$	$x_1^3x_4x_5x_7$	-197	197
$x_1^3x_4x_6x_7$	-512	2^9	$x_1^3x_5x_6x_7$	-463	463
	Subdivision	"$2+2+2+0+0+0+0$"			
$x_1^2x_2^2x_3^2$	-64	2^6	$x_1^2x_2^2x_4^2$	-512	2^9
$x_1^2x_2^2x_5^2$	-64	2^6	$x_1^2x_2^2x_6^2$	-512	2^9
$x_1^2x_3^2x_5^2$	-64	2^6			
	Subdivision	"$2+2+1+1+0+0+0$"			
$x_1^2x_2^2x_3x_4$	-43	43	$x_1^2x_2^2x_3x_5$	-71	71
$x_1^2x_2^2x_3x_6$	-113	113	$x_1^2x_2^2x_3x_7$	-1	
$x_1^2x_2^2x_4x_5$	-43	43	$x_1^2x_2^2x_4x_6$	-169	13^2
$x_1^2x_2^2x_4x_7$	-113	113	$x_1^2x_2^2x_5x_6$	-43	43
$x_1^2x_2^2x_5x_7$	-71	71	$x_1^2x_2^2x_6x_7$	-43	43
$x_1^2x_2x_3^2x_4$	-43	43	$x_1^2x_2x_3^2x_5$	-71	71
$x_1^2x_2x_3^2x_6$	-71	71	$x_1^2x_2x_3^2x_7$	-43	43
$x_1^2x_2x_3x_4^2$	-169	13^2	$x_1^2x_2x_3x_5^2$	-71	71
$x_1^2x_2x_3x_6^2$	-113	113	$x_1^2x_2x_4^2x_5$	-43	43
$x_1^2x_2x_4^2x_6$	-43	43	$x_1^2x_2x_4^2x_7$	-113	113
$x_1^2x_2x_4x_5^2$	-1		$x_1^2x_2x_4x_6^2$	-43	43
$x_1^2x_2x_5^2x_7$	-113	113	$x_1^2x_2x_5x_6^2$	-169	13^2
$x_1^2x_3^2x_5x_6$	-1		$x_1^2x_3^2x_5x_7$	-43	43
$x_1^2x_3^2x_6x_7$	-113	113	$x_1^2x_3x_4^2x_6$	-43	43
$x_1^2x_3x_4^2x_7$	-43	43	$x_1^2x_3x_4x_5^2$	-71	71
	Subdivision	"$2+1+1+1+1+0+0$"			
$x_1^2x_2x_3x_4x_5$	-29	29	$x_1^2x_2x_3x_4x_6$	-8	2^3
$x_1^2x_2x_3x_4x_7$	-8	2^3	$x_1^2x_2x_3x_5x_6$	-29	29
$x_1^2x_2x_3x_5x_7$	-29	29	$x_1^2x_2x_3x_6x_7$	-1	
$x_1^2x_2x_4x_5x_6$	-8	2^3	$x_1^2x_2x_4x_5x_7$	-1	
$x_1^2x_2x_4x_6x_7$	-29	29	$x_1^2x_2x_5x_6x_7$	-8	2^3
$x_1^2x_3x_4x_5x_6$	-1		$x_1^2x_3x_4x_5x_7$	-8	2^3
$x_1^2x_3x_4x_6x_7$	-29	29	$x_1^2x_3x_5x_6x_7$	-8	2^3
$x_1^2x_4x_5x_6x_7$	-29	29			
	Subdivision	"$1+1+1+1+1+1+0$"			
$x_1x_2x_3x_4x_5x_6$	-1				

$d = 2$ $n = 2^6$
$d = 3$ n is one of 2^3, 3^6, 43
$d = 4$ n is one of 2^{12}, 29, 71, 547
$d = 5$ n is one of 2^6, $2^3 \cdot 71$, 5^6, 13^2, $29 \cdot 113$, 43, 197, 421, 463
$d = 6$ n is one of 2^9, $2^6 \cdot 3^6$, $2^3 \cdot 29$, $2^6 \cdot 43$, 13^2, 29^2, $29 \cdot 449$, 41^2, $43 \cdot 71$, 113, 197, 211, 379, 463, 757, 2689.

It is also expected that the bound will decrease even further after the calculation of the elements of D suitable for every particular σ (omitted here). Once that calculation is done, the possible values of T can be constructed for each value of D, completing the algorithm.

For example, for degree 6, subdivision $6 + 0 + \cdots + 0$, the algorithm gives $n_\sigma = 6^6$. However, it is easy to see that calculating the suitable elements for D will show that in fact, the value of 6^1 would have been sufficient (in fact it is easy to see that G can be any subgroup of $(\mathbb{Z}_7 \times \mathbb{Z}_6^6) \rtimes \mathbb{S}_7$ and nothing else). Similar calculations can be done for most other subdivisions, but these need to be done on a case-by-case basis, and there does not seem to be a way of generalising them to any significant proportion of cases.

Bibliography

[1] A. Adler. On the automorphism group of a certain cubic threefold. *American Journal of Mathematics*, pages 1275–1280, 1978.

[2] H.F. Blichfeldt. *Finite collineation groups*. The University of Chicago Press, 1917.

[3] J.W. Bruce and C.T.C. Wall. On the classification of cubic surfaces. *J. London Math. Soc. (2)*, 19(2):245–256, 1979.

[4] H. Burkhardt. Untersuchungen aus dem gebiete der hyperelliptischen modulfunctionen. *Mathematische Annalen*, 38(2):161–224, 1891.

[5] H. Cartan. Quotient d'un espace analytique par un groupe d'automorphismes. In *Algebraic geometry and topology*, pages 90–102. Princeton University Press, Princeton, N. J., 1957. A symposium in honor of S. Lefschetz,.

[6] I. Cheltsov. Log canonical thresholds of del Pezzo surfaces. *Geometric and Functional Analysis*, 18(4):1118–1144, 2008.

[7] I. Cheltsov and C. Shramov. On exceptional quotient singularities. *Geom. Topol.*, 15(4):1843–1882, 2011.

[8] I. Cheltsov and C. Shramov. Six-dimensional exceptional quotient singularities. *Math. Res. Lett.*, 18(6):1121–1139, 2011.

[9] I. Cheltsov and C. Shramov. Weakly-exceptional singularities in higher dimensions. *Journal fur die Reine und Angewandte Mathematik*, to appear.

[10] C. Chevalley. Invariants of finite groups generated by reflections. *Amer. J. Math.*, 77:778–782, 1955.

[11] P.J. Davis. *Circulant matrices*. John Wiley & Sons, New York-Chichester-Brisbane, 1979. A Wiley-Interscience Publication, Pure and Applied Mathematics.

[12] J.-P. Demailly. On Tian's invariant and log canonical thresholds. An appendix to [**?**]. *Russian Mathematical Surveys*, 63(5):859, 2008.

[13] I.V. Dolgachev. McKay correspondence. Winter 2006/07. Lecture notes.

[14] I.V. Dolgachev and V.A. Iskovskikh. Finite subgroups of the plane Cremona group. In *Algebra, arithmetic, and geometry: in honor of Yu. I. Manin. Vol. I*, volume 269 of *Progr. Math.*, pages 443–548. Birkhäuser Boston Inc., Boston, MA, 2009.

[15] W. Feit. The current situation in the theory of finite simple groups. In *Actes du Congres International des Mathématiciens (Nice, 1970), Tome*, volume 1, pages 55–93, 1970.

[16] D.L. Flannery. The finite irreducible monomial linear groups of degree 4. *J. Algebra*, 218(2):436–469, 1999.

[17] F.G. Frobenius. *Über den von L. Bieberbach gefundenen Beweis eines Satzes von C. Jordan*. Königliche Akademie der Wissenschaften, 1911.

[18] V. González-Aguilera and A. Liendo. On the order of an automorphism of a smooth hypersurface. *arXiv:1102.3443*, 2011.

[19] B. Höfling. Finite irreducible imprimitive nonmonomial complex linear groups of degree 4. *J. Algebra*, 236(2):419–470, 2001.

[20] T. Hosoh. Automorphism groups of cubic surfaces. *J. Algebra*, 192(2):651–677, 1997.

[21] V.A. Iskovskikh and Y.G. Prokhorov. Fano varieties. In *Algebraic geometry, V*, volume 47 of *Encyclopaedia Math. Sci.*, pages 1–247. Springer, Berlin, 1999.

[22] J. Kollár. Singularities of pairs. In *Algebraic geometry—Santa Cruz 1995*, volume 62 of *Proc. Sympos. Pure Math.*, pages 221–287. Amer. Math. Soc., Providence, RI, 1997.

[23] S.A. Kudryavtsev. Pure log terminal blow-ups. *Mathematical Notes*, 69(5):814–819, 2001.

[24] T. Y. Lam and K. H. Leung. On vanishing sums of roots of unity. *J. Algebra*, 224(1):91–109, 2000.

[25] D. Markushevich and Y.G. Prokhorov. Exceptional quotient singularities. *American Journal of Mathematics*, 121(6):1179–1189, 1999.

[26] D. Mumford and G. Horrocks. A rank 2 vector bundle on $\mathbb{P}^4$ with 15,000 symmetries. *Topology*, 12:63–81, 1973.

[27] D. Prill. Local classification of quotients of complex manifolds by discontinuous groups. *Duke Math. J.*, 34:375–386, 1967.

[28] Y.G. Prokhorov. Blow-ups of canonical singularities. In *Algebra: proceedings of the International Algebraic Conference on the Occasion of the 90th Birthday of A.G. Kurosh, Moscow, Russia, May 25-30, 1998*, page 301. Walter De Gruyter Inc, 2000.

[29] Y.G. Prokhorov. Sparseness of exceptional quotient singularities. *Mathematical Notes*, 68(5):664–667, 2000.

[30] Yu. G. Prokhorov. Complements on conic fibrations. I. *J. Math. Sci. (New York)*, 106(5):3353–3364, 2001. Algebraic geometry, 11.

[31] D. Sakovics. Weakly-exceptional quotient singularities. *Cent. Eur. J. Math.*, 10(3):885–902, 2012.

[32] D. Sakovics. Five-dimensional weakly-exceptional quotient singularities. *Proc. of the Edinburgh Math. Soc.*, to appear.

[33] B. Segre. *The Non-singular Cubic Surfaces.* Oxford University Press, Oxford, 1942.

[34] Jean-Pierre Serre. *Linear representations of finite groups.* Springer-Verlag, New York, 1977. Translated from the second French edition by Leonard L. Scott, Graduate Texts in Mathematics, Vol. 42.

[35] G. C. Shephard and J. A. Todd. Finite unitary reflection groups. *Canadian J. Math.*, 6:274–304, 1954.

[36] V. V. Shokurov. Complements on surfaces. *J. Math. Sci. (New York)*, 102(2):3876–3932, 2000. Algebraic geometry, 10.

[37] V.V. Shokurov. 3-fold log flips. *Russ. Ac. SC Izv. Math.*, 40(1):95–202, 1993.

[38] T.A. Springer. *Invariant theory.* Springer-Verlag, 1977.

[39] J. Swallow. *Exploratory Galois theory.* Cambridge University Press, Cambridge, 2004.

[40] G. Tian. On Kähler-Einstein metrics on certain Kähler manifolds with $C\_1(M) > 0$. *Invent. Math.*, 89(2):225–246, 1987.

[41] G. Tian and S.-T. Yau. Kähler-Einstein metrics on complex surfaces with $C\_1 > 0$. *Comm. Math. Phys.*, 112(1):175–203, 1987.

[42] S.S.-T. Yau and Y. Yu. *Gorenstein quotient singularities in dimension three.* American Mathematical Society, Providence, RI, 1993.